NEW AXIOMS IN SET THEORY

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Abstract. In this article we review the present situation in the foundations of set theory, discussing two programs meant to overcome the phenomenon of independence centered, respectively, on forcing axioms and Woodin's V = Ultimate-L conjecture. While doing so, we briefly introduce the key notions of set theory.

Introduction

On January 6th, 1918, Georg Cantor passed away in Halle, after a life dedicated to the construction of the mathematical theory of infinity. After one hundred years his heritage is very much alive and set theory has reached a remarkable complexity of techniques and ideas. Not only set theory proved to be a useful tool in solving problems from different mathematical fields, but the conceptual sophistication of its development remained faithful to the philosophical importance of its creation.

Rendering infinity a trustful mathematical concept was not an easy task for Cantor [7], who was opposed on both mathematical and philosophical grounds by important intellectual figures, like Kronecker, who famously stated that "God created the natural numbers, all else is the work of man" [25], meaning that human understanding cannot go beyond the infinity of the collection of all natural numbers. However the progressive use of infinitary methods by mathematicians of the caliber of Weierstrass, Riemann and Dedekind paved the way to Cantor's mathematization of the infinite and his creation of set theory. Then, the true mathematical coronation happened in 1900 when Hilbert, during his famous Paris lecture [27], setting the mathematical agenda for the opening century, placed the solution of Cantor's Continuum Hypothesis as the first of his twenty-three problems[1]. Throughout his life Hilbert remained a strong supporter of Cantor's ideas, promising, in his famous 1925 paper *On the infinite*, that: "No one shall expel us from the paradise which Cantor has created for us".

The development of the mathematical theory of infinity was not only a story of success. In 1901, the discovery of Russell's paradox shook at the very base the entire edifice of mathematics, casting shadows on the coherence and thus the importance of infinity in mathematics. What became clear after that discovery was the existing gap between the rules that govern the infinite and those that govern the finite. Therefore it became imperative to

[1]For a well detailed history of set theory see [16].

lay down the right coherent laws that determine a correct use of infinity in mathematics.

In 1908 Zermelo proposed a careful axiomatization of set theory. Following Hilbert's enthusiasm in Cantor's theory —and in the attempt to justify his use of the Axiom of Choice AC in the proof of the Well-ordering Theorem— he introduced a first list of axioms. After the contributions of Fraenkel and von Neumann, this list became the standard axiomatization of set theory ZFC. By means of ZFC it was possible to avoid the known paradoxes, but nonetheless many fundamental questions remained opened for decades; among those Cantor's Continuum Hypothesis CH.

The first fundamental step in the attempt to give a definitive answer to CH was published in 1938, when Gödel [22] showed its (relative) consistency with the axioms of ZFC. Almost thirty years later, in 1963, the decidability of CH on the basis of the ZFC-axioms received a negative answer, when Cohen [9] established the relative consistency of the failure of CH with ZFC. Gödel and Cohen's result showed that CH is undecidable, that is —rephrased with the proper terminology of mathematical logic— provably independent with respect to the ZFC-axioms.

Cohen's fundamental breakthrough, worth a Fields medal, opened a new era in set theory, both technically and conceptually. The sophistication of Cohen's technique allowed to show the undecidability (i.e. independence relative to ZFC) of many problems remained opened until then. All these results had the effect of starting a major discussion on the limits of axiomatization and on the sharpness of set theoretical concepts, among which that of infinity.

In this article we review some of the strategies to overcome the intrinsic limitations of ZFC. Without any ambition of completeness[2], we will present the so-called Gödel's program, its extension in terms of generic absoluteness for second order arithmetic, and its ramifications given, respectively, by forcing axioms, and by Woodin's program centered around the construction of the so-called Ultimate-L. Before that, we briefly introduce the main definitions and ideas which guided the development of set theory during the last one and a half century.

The paper is organized as follows: §1 introduces the basic set theoretic concepts, §2 deals with Cantor's notion of cardinality and of well-order, §3 deals with the concept of undecidability in mathematics, §4 presents the strategy of Gödel to overcome the independence phenomenon in set theory, §5 gives a brief account of the implications of large cardinal axioms on second order arithmetic, §6 briefly discusses the two research programs meant to overcome the undecidability of CH.

There is a certain overlap between §1 and §2 of the present paper and §2, §3, and §4 of Andretta's article in this issue. The reader of both articles can

[2]See [56] for a bird-eye view on all approaches that can be found in the literature.

skim through these sections in either of them or instead take advantage of the two distinct presentations of these topics.

We tried to make (most of) the paper accessible to readers with a good training in mathematics at the level of a bachelor degree in mathematics. We also tried to keep the prerequisites in logic and in set theory to a minimum. It is our hope that this is the case for the content of §1, §2, §3 (with the exception of §3.3), §4. Nonetheless we are aware that those who have some background in logic and/or set theory will greatly benefit of it during their reading. §5 and §6 will require a steadily increasing familiarity with delicate and technical set theoretic notions (even though we hope that this does not preclude the non-expert reader to get the main ideas presented in §5). On the other hand §3.3 demands a big effort on the reader, and has the aim of presenting measurable functions as the non-standard instantiation of the concept of real number given by a certain kind of forcing notion. Familiarity with the basics of first order logic will simplify this effort; however the reader may safely skip all of §3.3 without compromising the comprehension of the subsequent parts of the paper.

1. The universe of all sets

In the rapid development of set theory following the initial astonishing results of Cantor, it became clear that not only this theory could give a mathematically precise formulation and justification of the infinitary methods already widespread in algebra and analysis, but it could also offer a common framework where to develop all known mathematics [42]. Indeed, the simple and abstract language of set theory is so versatile that almost every mathematical structure can be therein defined and proved to exist. This peculiar character of set theory is what is normally called the *foundational* role of set theory, or, in less ontological terms, its *universality.*

1.1. **Russell's paradox.**

The process that led to the formalization of set theory is full of trials and errors, with successive efforts attempting to caliber the right expressive power of the theory.

An uncritical attitude towards the laws of logic, at the end of the XIX century, induced Frege and Dedekind, among many others, to believe that it was always possible to define a set in terms of a property shared by all its members; that is, given any well defined property $\varphi(x)$, it is always possible to form the set $\{x : \varphi(x)\}$ which is the family of elements x satisfying the property $\varphi(x)$. This led Frege to propose a foundations of mathematics entirely based on logic. Unfortunately, this approach was shown to be inconsistent by Russell, in 1901. Russell's (in)famous paradox, states that the set of all objects that do not belong to themselves — which is itself a well defined property — cannot exist. This is easily shown by the following argument. Define $R = \{x : x \notin x\}$ (where $x \in y$ stands for "x is an element of y"). Assume R is a set, then either $R \in R$ or $R \notin R$. If $R \in R$, it satisfies its defining property, i.e. it is a set x which does not belong to itself, yielding that $R \notin R$. By a similar argument we can also infer that if $R \notin R$, then $R \in R$. Thus we get that $R \in R$ if and only if $R \notin R$: a contradiction.

1.2. **The ZFC-axioms.** In order to amend set theory from its paradoxical consequences, Zermelo decided to collect a list of axioms strong enough to develop the results obtained by Cantor, but weak enough to avoid any paradox [64]. Zermelo's list was later improved by the contributions of Fraenkel and von Neumann to form the axiom system ZFC.

The basic idea guiding Zermelo's list of axioms is the following: we need existence axioms asserting that certain sets, like the natural numbers $\mathbb{N}$, exist and construction principles able to build new sets from previously given ones; in order, for example, to construct $\mathbb{R}$ from $\mathbb{N}$. On the other hand, Zermelo's list of axioms needs to be weak enough to avoid $R = \{x : \varphi(x)\}$ being a set; otherwise Russell's paradox would apply, and set theory, as formalized by these axioms, would be inconsistent[3]. The axioms of ZFC are the following[4].

Extensionality: *Two sets are equal if and only if they have the same elements.*

This occurs regardless of how the sets are defined, or of the order by which their elements are presented. Therefore we get, for example, the following equalities:

$$\{x : x \in \mathbb{Z} \text{ and } x^2 - 3x + 2 = 0\} = \{1, 2\} = \{2, 1\}.$$

The above equations show that two sets defined in terms of two different properties (namely being an integer solution of the equation $x^2 - 3x + 2 = 0$ or being equal to 1 or 2) are the same, simply because they have the same elements (moreover the listing of the elements is irrelevant to decide an equality, as $\{1, 2\} = \{2, 1\}$).

Empty-set: *There exists a set with no elements.*

This axiom grants that the universe of sets contains something. The empty set is unique: by extensionality there cannot be two distinct sets with no element. It is customary to denote with $\emptyset$ the empty-set.

Pairing axiom: If X, Y are sets, so is $\{X, Y\}$.

Union axiom: If X is a set, so is $\bigcup X = \{z : z \in y \text{ for some } y \in X\}$.

The pairing axom and the union axiom are basic construction principles already sufficient to construct several important sets of finite size starting from the empty set. Notice that the usual binary union $X \cup Y$ can be defined as $\bigcup\{X, Y\}$.

[3] A nice reference text containing the fundamental results of set theory is (among many others) [28].

[4] Modulo the unavoidable inaccuracies due to the fact that we sidestep (for lack of space) the use of first order logic in our presentation of formal systems.

Separation: *If X is a set and $\phi(x)$ is a well-defined property*[5] $\{x \in X : \phi(x)\}$ *is also a set.*

Even though quite similar to Frege's comprehension principle (asserting that $\{x : \phi(x)\}$ is a set for every property $\phi(x)$), Separation imposes a fundamental restriction on where the elements of the new set come from: namely they are elements of a set X. Intuitively this means that, being a set, X transfers to its subsets a label of trustfulness for non-contradictory objects. Indeed Separation is weak enough to avoid any proof of Russell's paradox from the ZFC-axioms.

Power set: *If X is a set then $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ (the set of all subsets of X) is also a set.*

Existence of the natural numbers: *There exists an infinite set*[6].

This axiom is necessary: it is possible to construct a universe of sets in which all sets are finite (i.e. this axiom does not hold in the model), while all the other axioms of ZFC are satisfied (see §5.1).

Choice (AC): *If $\{X_i : i \in I\}$ is a non-empty set and each X_i is non-empty for all $i \in I$, we also have that $\prod_{i\in I} X_i$ is non-empty (i.e. there is some $f : I \to \bigcup\{X_i : i \in I\}$ such that $f(i) \in X_i$ for all $i \in I$).*

If I is finite one does not need AC to find an element in the product $\prod_{i\in I} X_i$. But assume for example x is an accumulation point of some set $A \subseteq \mathbb{R}$: how do we select inside A a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x? AC shows that this is possible by choosing an element of the product $\prod_{i\in\mathbb{N}}(A \cap B_{\frac{1}{i+1}}(x))$ (the latter is the ball in x of radius $\frac{1}{i+1}$). There are sets $A \subseteq \mathbb{R}$ having accumulation points for which we cannot prove that such a sequence can be found without appealing to AC.

Replacement: *If X is a set and $F(x, y)$ is a functional property (i.e. a property for which one can prove that for all sets x there is only one set y such that $F(x, y)$ holds), then $F[X] = \{y : \exists x \in X\, F(x, y)\}$ (the pointwise image of X by F) is also a set.*

Foundation: *The binary relation $\in$ is well-founded, i.e. there is no infinite chain $\{x_n : n \in \mathbb{N}\}$ such that $x_{n+1} \in x_n$ for all n.*

[5] It is rather delicate to define properly what it means for a property to be well defined: the precise definition is that of being expressible in the first order language with a binary relation symbol for the $\in$-relation; see [34] for more details. We omit any further discussion of this matter here.

[6] More details on the correct formulation of this axiom will be given in section 2.3.

Foundation and Replacement come from the contributions of, respectively, von Neumann and Fraenkel, the others are modern formulations of the original axioms of Zermelo. Foundation and Replacement are useful in dealing with large classes of structures. They are extremely useful to develop the structural theory of the universe of sets and replacement is necessary to formalize properly category theory, or any mathematical field where one deals with a large family of mathematical objects at the same time (groups, rings, etc.).

It should be noted that almost all of mathematics can be developed by means of Zermelo's axioms. For example the reals, the complex numbers, the L^p-spaces, and many other objects from number theory, differential and algebraic geometry, functional analysis, general topology, etc. can be proved to exist on the basis of the ZFC-axioms. Moreover almost all the relevant theorems about these structures are logical consequences of these principles.

The picture of the universe of sets offered by Zermelo's axioms has a conceptual cost: that of introducing collections that are not sets. Indeed, the collection of all sets $V = \{x : x = x\}$ is not a set, otherwise $R = \{x \in V : x \notin x\}$ would also be one, by the separation axiom applied to V and the property $x \notin x$, leading thus to Russell's paradox. This conceptual cost is exactly the leverage that permits to avoid all known paradoxes, like Russell's. We only need to accept that there are collections of sets that are too big to be considered sets (such as $R = \{x \in V : x \notin x\}$); these collections are normally called *proper classes*. Some care must be paid in handling correctly the distinctions between proper classes and sets. We will come back on this topic at the end of section 2.

2. Ordinals, cardinals, and the structure of the universe of sets

In 1895 Cantor published a first summary of the most important results of set theory [8]. There he explained the origin of the two main concepts of his theory: that of ordinal number and that of cardinal number.

When we conceive a collection of mathematical objects, we may abstract from all of their peculiar properties except for the ordering by which they present themselves to our mind; the "well-order" by which these objects are organized gives raise to the concept of ordinal type of a collection; we can even abstract from the well-order of a collection and retain only its "quantity". This second notion gives rise to the concept of cardinal number.

2.1. **Natural numbers.** Let us first deal with the formalization of the concept of natural number in set theory. Natural numbers are at the same time the simplest examples of ordinals and cardinals, hence it is "natural" to start our analysis of Cantor's notion of cardinality and well-orders briefly describing how these numbers can be represented by sets.

It is natural to define $\mathsf{0}$ to be the empty-set, the latter being the unique set with zero elements. Among the variety of sets containing just one element

$\{\mathsf{0}\} = \{\emptyset\} = \{\emptyset, \emptyset\}$ can be defined appealing to the pairing axiom, the extensionality axiom, and the fact that we know what is $\mathsf{0}$. Let us call this set $\mathsf{1}$. Similarly we can define $\mathsf{2} = \{\mathsf{0}, \mathsf{1}\}$ taking the set whose elements are $\mathsf{0}$ and $\mathsf{1}$, and observing that it has exactly two elements; proceeding so on so forth, we can define the sets $\mathsf{n} = \{\mathsf{0}, \ldots, \mathsf{n} - \mathsf{1}\}$ consisting of all its preceding numbers, and observe that it has exactly n-many elements. To define these sets we just need to appeal to the axiom of extensionality (to grant their uniqueness), the empty-set axiom (to grant existence of $\mathsf{0}$), and to the pairing and union axioms, since for all n:

$$\mathsf{n} = \mathsf{n} - \mathsf{1} \cup \{\mathsf{n} - \mathsf{1}\} = \cup\{\mathsf{n} - \mathsf{1}, \{\mathsf{n} - \mathsf{1}\}\}.$$

The current ZFC-formulation of the axiom of infinity states that there exists a set containing all the n, more precisely it states that:

> *There exists a set X such that $\emptyset \in X$ and if $z \in X$ also $z \cup \{z\} \in X$.*

The set of natural number $\mathbb{N}$ can be defined as the subset of X whose elements are exactly all the sets n. It is a bit delicate to show by means of the separation axiom, the power-set axiom, and the axiom of infinity that $\mathbb{N}$ exists. We skip the details.

Similarly one can prove on the basis of the ZFC-axioms that the rationals, the integers, the reals, the complex numbers etc. are (representable by) sets. The key point is that the usual textbooks of algebra or calculus build these objects starting from the natural numbers — which we know to be (representable by) a set — and employ at various stages construction principles which are easily derivable from the ZFC-axioms. We will come back to this point later on in section 2.6. For the moment we skip the details and assume that all the above objects are elements of the universe of sets.

2.2. **Cardinal numbers.** Cardinal numbers represent the cornerstone of Cantor's theory of infinity. In modern notation, given two sets X, Y, we write $X \approx Y$ whenever there is a bijection $f : X \to Y$. This defines an equivalence relation on the collection of all sets. It is customary to denote the cardinality of a set X (i.e. its equivalence class according to $\approx$) by $|X|$. Dealing with the relation $\approx$ is rather delicate, since it can be proved that $|X|$ is not a set for any non-empty set X; otherwise we would run into a paradox. Nonetheless, using the theory of ordinal numbers we will sketch in 2.3, one can select a canonical representative of each equivalence class $|X|$, its *cardinality*, or its *cardinal number*.

Moreover we have already introduced canonical representatives of finite sets i.e. the sets n, which we can use to define which are the finite sets: a set X is finite if it is in bijection with some n; infinite if it is not finite. Other important classes are also the countable sets, i.e. those in bijection with $\mathbb{N}$, which define the cardinality class of $\mathbb{N}$, and the uncountable sets, which are neither finite, nor countable.

Cantor defines the notions of sum, product, and exponentiation of arbitrary cardinalities as follows:

$|X| + |Y| = |X \times \{\mathsf{0}\} \cup Y \times \{\mathsf{1}\}|$,
$|X| \cdot |Y| = |X \times Y|$,
$|X|^{|Y|} = |X^Y|$ where $X^Y = \{f : f$ is a function from Y to $X\}$.

These rules are well defined and generalize to cardinalities the usual arithmetical operations on natural numbers. Indeed $\mathsf{n} \times \mathsf{m}$ is in bijection with k, with $k = n \cdot m$, n^{m} —i.e. the set of functions $f : \mathsf{m} \to \mathsf{n}$— is in bijection with l, with $l = n^m$, and $\mathsf{m} \times \{0\} \cup \mathsf{n} \times \{1\}$ is in bijection with j, with $j = m + n$.

It is natural to define an order relation between cardinalities, saying that X is not larger than Y ($|X| \leq |Y|$), if there is an injective function $f : X \to Y$. As before this property holds for finite sets, since for example the set $\{a_0, \ldots, a_{n-1}\}$ can be injected into $\{b_0, \ldots, b_{m-1}\}$ if and only if $n \leq m$. An important property of this order relation, now bearing the name of Cantor-Schröder-Bernstein Theorem (CSB in what follows), states that $|X| = |Y|$ if and only if $|X| \leq |Y| \leq |X|$.[7]

So far there seems to be a complete accordance between the laws of set theory and those of arithmetics, but unexpected surprises appear when we consider the case of infinite cardinalities. Indeed, one of the first results of Cantor shows that $|X| + |Y| = |X| \cdot |Y| = \max\{|X|, |Y|\}$, whenever at least one, among X and Y, is infinite. These first results came together with the discovery of unexpected (and in some cases astonishing) equalities and differences between cardinalities of familiar infinite sets. To the first kind belongs the fact that $|\mathbb{Q}| = |\mathbb{N}|$, or even that there are as many algebraic numbers (i.e. solutions of polynomials in one variable with integer coefficients) as natural numbers.

In the case of rational and natural numbers it is clear that $|\mathbb{N}| \leq |\mathbb{Q}|$, as witnessed by the identity function. On the other hand $|\mathbb{Q}| \leq |\mathbb{N}|$ can be proved as follows: First of all $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}|$, since $\mathbb{Q}$ can be identified with the subset of $\mathbb{Z} \times \mathbb{N}$ given by pairs (n, m), with n, m coprimes and $m > 0$. It is also easy to define a bijection of $\mathbb{Z} \times \mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$; for example there is a bijection ϕ between $\mathbb{Z}$ and $\mathbb{N}$ given by $\phi : n \mapsto -2n$ if $n \leq 0$ and $\phi : n \mapsto 2n + 1$ otherwise. Then $\phi \times \mathrm{Id} : (n, m) \mapsto (\phi(n), m)$ witnesses $|\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Finally the map $(m, n) \mapsto 2^{m+1} \cdot 3^{n+1}$ is an injection witnessing that $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$. Composing all these (in)equalities we obtain:

$$|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|;$$

[7] One direction of this equivalence is trivial since if $f : X \to Y$ is a bijection then f, f^{-1} are both injections. On the other hand if we know that $f : X \to Y$ and $h : Y \to X$ are injections, it is not at all clear how to find a bijective $g : X \to Y$. Consider for example the continuous injection $f : [0; 1] \to (0; 1)$ given by $x \mapsto \frac{1}{3} + \frac{x}{2}$, let $h : (0; 1) \to [0; 1]$ be the inclusion map. These maps witness that $|[0; 1]| \leq |(0; 1)| \leq |[0; 1]|$. On the other hand any bijection between $[0; 1]$ and $(0; 1)$ cannot be continuous, since the two spaces are not homeomorphic, while the injections defined above are both continuous. But CSB entails that a (non-continuous) bijection can be found.

we conclude that $|\mathbb{Q}| = |\mathbb{N}|$ by CSB.

With the same line of reasoning, using a smarter codification of all n-tuples of natural numbers with natural numbers, it is possible to define an injective map from the algebraic numbers into $\mathbb{N}$. By CSB we conclude that the two sets are in bijection.

The first true coronation of the theory of cardinal numbers is, however, the discovery of the existence of infinitely many infinite cardinalities. One of Cantor's most celebrated theorem states that $|X| < |\mathcal{P}(X)|$ for all sets X. Its proof is obtained by a smart tweak of Russell's paradox: suppose, by way of contradiction, that there is a bijective function $f : \mathcal{P}(X) \to X$ and define $Y = \{z \in X : z \notin f^{-1}(z)\}$. Now, being $Y \subseteq X$, we can apply f to it and ask whether $f(Y) \in Y$. If this were the case, then, by definition of Y, we would have $f(Y) \notin f^{-1}(f(Y)) = Y$. On the other hand if $f(Y) \notin Y$, by a similar argument, we can infer $f(Y) \in Y$, thus showing that $f(Y) \in Y$ if and only if $f(Y) \notin Y$, a contradiction. The map $x \mapsto \{x\}$ defines an injection of X into $\mathcal{P}(X)$, hence $|X| < |\mathcal{P}(X)|$. Moreover, by iterating the powerset operation, we get larger and larger infinite cardinalities. It can also be shown that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$. Hence by Cantor's Theorem $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

We now would like to give a better picture of how cardinals are organized by the order relation $|X| \leq |Y|$. Indeed, with the information at disposal so far it is not even clear that the order $|X| \leq |Y|$ is linear. It is time to introduce the second key notion of set theory.

2.3. **Ordinal numbers.** $(X, <)$ is a well-order if $<$ is a linear order on X such that any non-empty subset of X has a least element according to $<$. For example, every finite linear order is a well-order, and the natural numbers with the usual order (i.e. the structure $(\mathbb{N}, <)$) is a well-order[8]. But there are a huge quantity of infinite well-orders which are not isomorphic to $(\mathbb{N}, <)$: A first example is given by $(\mathbb{N} \times \mathbb{N}, <_{\mathrm{lex}})$ where $(m, n) <_{\mathrm{lex}} (p, q)$ if either[9] $m < p$ or ($m = p$ and $n < q$). This well-order is not isomorphic to $(\mathbb{N}, <)$ since $(1, 0)$ is bigger than $(0, m)$, for all $m \in \mathbb{N}$.

A more sophisticated example is given by the linear order $(\mathbb{N}[x], <)$, where $p(x) = \sum_{i=0}^{n} a_i x^i < \sum_{i=0}^{m} b_i x^i = q(x)$ if either $(n < m)$ or, ($n = m$ and $a_i < b_i$ for the least i such that $a_i \neq b_i$). Notice that this order has the monomial x^n above all polynomials of degree less than n, and $(\mathbb{N} \times \mathbb{N}, <_{\mathrm{lex}})$ can be identified inside $(\mathbb{N}[x], <)$ as the set of polynomials of degree at most 1, i.e. the predecessors of x^2.

A fundamental result on well-orders, due to Zermelo, is the Well-Ordering Theorem, which is actually an equivalent formulation of AC. It states that for every non-empty set X there is at least one binary relation $<$ on X such that $(X, <)$ is a well-order.

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[8] The natural numbers can be characterized as the unique infinite well-order whose upward bounded subsets have a maximum. This provides an equivalent formulation of the induction principle on $\mathbb{N}$.

[9] $\leq_{\mathrm{lex}}$ is the usual lexicographic order on $\mathbb{N}^2$.

A second fundamental result on well-orders is due to von Neumann (and Mostowski). To formulate it we need to introduce the notion of *transitive set*: a set X is transitive if for all $a \in X$, $a \subseteq X$.

We say that a set α is a *(von Neumann) ordinal* if it is transitive and such that $(\alpha, \in)$ is a linear order. We invite the reader to check that the sets $\mathsf{0}, \mathsf{1}, \ldots, \mathsf{4}, \ldots$ are transitive. It can be proved that every n is a transitive set (and other examples of interesting transitive sets will come in due course).

von Neumann (following a more general result by Mostowski) proves that for each well-order $(X, <)$ there is a unique von Neumann ordinal α and a unique order preserving bijection $f : X \to \alpha$.

von Neumann ordinals provide canonical representatives of the isomorphism types $\underline{X}$ of a well-order $(X, <)$, i.e. the unique transitive set α such that $(\alpha, \in)$ belongs to $\underline{X}$.

For example von Neumann theorem entails that any finite linear order is isomorphic to $(\mathsf{n}, \in)$, for some unique n and the isomorphism is unique.

However the full strength of von Neumann's theorem shows up when we have to deal with well orders more complicated than $\mathbb{N}$. For example we get that there are unique ordinals α, β such that $(\mathbb{N} \times \mathbb{N}, <_{\text{lex}})$ is isomorphic to $(\alpha, \in)$ and $(\mathbb{N}[x], <)$ is isomorphic to $(\beta, \in)$. Moreover one can check that $\alpha \in \beta$ holds as well.

A third fundamental result on well orders, due to Cantor, asserts that for any two well orders $(X, <_X)$ $(Y, <_Y)$ there is a trichotomy: either there is a unique order preserving bijection $f : X \to Y$, i.e. $(Y, <_Y)$ is isomorphic to $(X, <_X)$; or there is some $a \in Y$ and a unique order preserving $f : X \to Y$ such that $f[X] = \{b \in Y : b <_Y a\}$, that is $(X, <_X)$ is isomorphic to an initial segment of $(Y, <_Y)$; or conversely $(Y, <_Y)$ is uniquely isomorphic to an initial segment of $(X, <_X)$.

One can use Cantor's trichotomy theorem, von Neumann's theorem and some extra work to show that for any two von Neumann ordinals α, β, either $\alpha \in \beta$ (when $(\alpha, \in)$ is isomorphic to an initial segment of $(\beta, \in)$), or $\beta \in \alpha$, or $\alpha = \beta$. Therefore one can compare the isomorphism types of two well-orders simply checking whether the unique ordinals in these isomorphism types belong to one another or are equal.

For ordinals α, β, we will write $\alpha \leq \beta$ to denote that $\alpha \in \beta$ or $\alpha = \beta$. The class Ord, given by von Neumann ordinals, is itsef well ordered by $\in$: if $C \subseteq \mathsf{Ord}$ is non-empty, take $\beta \in C$. If $C \cap \beta$ is empty then $\beta = \min C$; otherwise $C \cap \beta \subseteq \beta$ is non-empty, hence –since $(\beta, \in)$ is a well-order– it has an $\in$-minimal element α, giving that $\alpha = \min(C)$.

2.4. **Many infinities, but which?** Remark that for ordinals α, β, $\alpha \in \beta$ entails that $\alpha \subseteq \beta$ (by transitivity of β), which trivially gives that $|\alpha| \leq |\beta|$. von Neumann's theorem applied to the well-orders $(\mathbb{N}, <), (\mathbb{N} \times \mathbb{N}, <_{\text{lex}})$, and $(\mathbb{N}[x], <)$ gives us three distinct ordinals: $\omega \neq \alpha \neq \beta$. By Cantor's trichotomy we also have that $\omega \in \alpha \in \beta$. On the other hand it is not hard to check that $|\omega| = |\alpha| = |\beta|$; it also holds that $|\omega| > |\mathsf{n}|$ for all $\mathsf{n} \in \omega$;

actually ω is the least ordinal (according to $\in$) in the cardinality class of countable sets.

More generally by means of Zermelo's Well-ordering Theorem one gets that $\mathsf{Ord} \cap |X|$ is a non-empty subset of Ord for any non-empty cardinality class $|X|$. Hence each cardinality class $|X|$ must have an $\in$-minimal ordinal, which we consider the canonical representative of $|X|$.

We are now in the position to define the subclass Card of Ord, given by these canonical representatives of all cardinalities[10]:

$$\mathsf{Card} = \{\kappa : \kappa \text{ is the } \in\text{-least ordinal in } |\kappa|\}.$$

By the trichotomy of ordinals, from $\lambda \neq \kappa \in \mathsf{Card}$ we get that either $\lambda \in \kappa$ or $\kappa \in \lambda$. Assuming the first, since κ is transitive, we get that $\lambda \subseteq \kappa$, hence $|\lambda| \leq |\kappa|$. But since $\lambda \notin |\kappa|$, being two different cardinals, we conclude that $|\lambda| < |\kappa|$. Therefore we have just proved the following.

For $\kappa \neq \lambda \in \mathsf{Card}$, $|\lambda| < |\kappa|$ if and only if $\lambda \in \kappa$.

Hence the $\in$-relation restricted to Card well-orders its elements according to their cardinality, moreover the sets n are the canonical representatives in Card of the cardinality classes of finite sets. It is customary to denote by $\aleph_\alpha$ the α-th infinite element of Card according to its well-order given by $\in$ and to confuse a cardinality class $|X|$ with the unique $\aleph_\alpha \in |X|$. With this notation $|\omega| = \aleph_0$, $\aleph_1$ is the least uncountable cardinal, $\aleph_2$ the second uncountable cardinal, etc.

2.5. **Cantor's continuum problem.** We have a very simple table of sum and multiplication for infinite cardinalities, as well as a nice ordering between cardinalities (it is a proper class that can be well ordered by a well order of length Ord). It is now time to address the table of the exponential map $\kappa \mapsto 2^\kappa$.

Cantor's continuum hypothesis CH can be formulated without any reference to well-orders as follows:

$$(\mathsf{CH}_0) \qquad \text{There is no set } X \text{ such that } |\mathbb{N}| < |X| < |\mathcal{P}(\mathbb{N})|.$$

If we take into account the general theory of cardinalities so far presented, and recall that $2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, we can rephrase CH as

$$(\mathsf{CH}_1) \qquad 2^{\aleph_0} = \aleph_1.$$

The attempt to verify or falsify CH represents a common trait of a large portion of the history of set theory, and it is indeed the attempt of finding a satisfactory solution to the question of how many real numbers there are, that motivates and explains the two programs we will describe at the end of this paper.

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[10] For thse familiar with first order logic the definition of Card is slightly problematic: it uses the collection of proper classes given by cardinalities to define a new proper class, it is not transparent that with such a definition Card is the extension of a well-defined property according to the first order formalization of ZFC. With some work (which we omit) it can be shown that this is indeed the case,

2.6. **The cumulative hierarchy.** To proceed further, it is in our eyes advisable to step back a bit and outline Zermelo's work on the structure of the universe of sets.

In order to secure the consistency of ZFC, in 1930 Zermelo [65] was able to give a precise description of the structure of the universe of sets according to the ZFC-axioms. The simplest analogy with Zermelo's work comes from a comparison of ZFC with the Peano's axiomatization of natural numbers. $\mathbb{N}$ can be intuitively described as the structure obtained iterating the successor operator $n \mapsto n+1$ starting from the element 0 and generating one after the other the natural numbers, $0, 1, 2, 3, \ldots, n, n+1, \ldots$. Peano's axioms describe $\mathbb{N}$ by giving details on the properties of the successor operation and of its interaction with the operations of sum and multiplication. Moreover the structure $(\mathbb{N}, +, \cdot, 0, 1, =)$ is the "standard model" of Peano axioms for arithmetics[11].

Zermelo's work took the same approach with the ambition to give a clear intuitive picture of the mathematical structure whose elements are all and only sets: in modern terminology the *universe of all sets*.

As $\mathbb{N}$ can be obtained iterating the successor operator $n \mapsto n+1$, a similar iterative conception was shown to be essential for the notion of set. Zermelo showed that the axioms of ZFC grant that the universe of all sets is stratified in a *cumulative hierarchy*, where one now uses the power-set operation to generate the new elements of the hierarchy. In this case we can say that the ZFC-axioms capture the key properties of the power-set operation describing its interactions with other simpler set-construction principles.

To describe Zermelo's stratification of the universe of sets it is useful to refresh some piece of notation. The symbol ω indicates the well-order of $\mathbb{N}$, while $\omega + n$ is the well-order obtained putting a (the) linear order of n-elements on top of the natural numbers. Now we can stratify the universe of sets as follows:

$V_0 = \emptyset = \mathsf{0}$,
$V_1 = \mathcal{P}(V_0) = \{\emptyset\} = \mathsf{1}$,
$V_2 = \mathcal{P}(V_1) = \mathcal{P}(\mathcal{P}(V_0)) = \{\emptyset, \{\emptyset\}\} = \mathsf{2}$,
$V_3 = \mathcal{P}(V_2) = \mathcal{P}(\mathcal{P}(V_1)) = \mathcal{P}(\mathcal{P}(\mathcal{P}(V_0))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ $(\neq \mathsf{3})$,
......
......
$V_\omega = \bigcup_{n \in \mathbb{N}} V_n$,
$V_{\omega+1} = \mathcal{P}(V_\omega)$,
$V_{\omega+2} = \mathcal{P}(\mathcal{P}(V_\omega))$,
.....

[11] See §3.1 for a brief description of the notion of model of a family of axioms.

By iterating transfinitely this operation for all ordinals α, one can define V_α the collection of all sets we can produce in α-many steps[12].

In the structure $(V_{\omega+1}, \in, =)$ one can already develop number theory, most of analysis, and great parts of areas of mathematics like differential geometry; while functional analysis deals mainly with objects belonging to (or at least subsets of) $V_{\omega+2}$.

Recalling that Ord is the collection of all von Neumann ordinals, the universe of all sets V is provably given by the union of all the V_α, i.e.:

$$V = \bigcup_{\alpha \in \mathsf{Ord}} V_\alpha.$$

2.7. **On the distinction between sets and proper classes.** Some practical hints to handle correctly sets and proper classes are the following:

- Almost all interesting mathematical entities are (represented by) sets.
- There are interesting collections of sets which are not themselves sets. These families of sets are named *proper classes*. Examples are: the family $V = \{x : x = x\}$ (where the variable x ranges over all sets) which is the largest proper class consisting of all sets; the family of all groups; the family of all topological spaces; and many other families, for example it can be shown that Card and Ord are as well proper classes.
- All elements of a set or of a proper class are themselves sets, indeed any set or class is contained in $V = \{x : x = x\}$, which is therefore transitive. As a matter of fact, if $X \in V$, then $X \subseteq V$: since X is a set, all elements of X are themselves sets.
- The key distinction between a set and a proper class is that a set X always belong to a proper class, as $X \in V$ for any set X, while proper classes are exactly those collections of sets which are "too large" to belong to some set, on the pain of running into paradoxes.
- With the provision that a proper class is not an element of a set, one can safely handle proper classes much in the same way one handles sets. For example the union $\bigcup_{i \in I} C_i$ of an indexed family of proper classes is itself a proper class. In a similar way, the intersection of proper classes is a perfectly defined entity that can be wither a proper class, or a set. On the other hand if C is a proper class its powerset $\mathcal{P}(C)$ does not exist: if $\mathcal{P}(C)$ were a proper class, its elements would be sets, but C is an element of $\mathcal{P}(C)$ and is not a set, being a proper class.

It takes some practice to understand which families of sets cannot be sets, which set-theoretic operations, such as unions, intersections, or sets sized

[12]It can be proved that V_α is transitive for all α, but not linearly orderd by $\in$ for $n > 2$; we invite the reader to check that this is the case for the first V_ns.

products, can be safely performed on a (family of) proper class(es), yielding perfectly well-defined collections, and which set-theoretic operations are not valid for proper classes, as the power-set operation, for they lead to the construction of paradoxical entities. However, the practical knowledge acquired since Cantor made the paradoxes arising from an improper use of the concept of set and proper class not anymore a source of concerns for working mathematicians and set theorists.

3. The phenomenon of independence

Once a sufficiently clear picture of the universe of sets was available, the belief in the possibility to find a definitive answer to the most pressing set theoretical questions became stronger. Put otherwise, after Zermelo's definition of V, most mathematicians held the view that it was only a matter of time (and hard work) to find the solution of questions like CH.

3.1. The independence of CH.

For many decades Cantor and many others attempted to prove or disprove CH. But it took the genius of both Gödel and Cohen to show that all these attempts were doomed to fail, at least in the context of the ZFC-axioms. Indeed, in two subsequent and complementary steps, Gödel [22] in 1938, and Cohen [9] in 1963, showed that, just on the basis of the ZFC-axioms, it is impossible, respectively, to prove the negation of CH, or to prove CH.

We briefly introduce the notion of formal independence, a necessary step in order to grasp the content of Gödel and Cohen's result.

Let Γ be a bunch of mathematical assertions, for example the group axioms. These assertions may, or may not, correctly describe the properties of a given mathematical structure $\mathcal{M}$. In case the statements in Γ assert true facts about $\mathcal{M}$, we say that $\mathcal{M}$ is a model of Γ, or that Γ is valid in (or satisfied by) $\mathcal{M}$, and write $\mathcal{M} \models \Gamma$.

The precise definitions of these concepts would need a lenghty detour in first order logic; one we will not take for the sake of brevity. Instead, let us exemplify these notions with an example.

The group axioms are the following statements:

$$\forall x \forall y \forall z\ (x{*}y){*}z = x{*}(y{*}z), \quad \forall y\ (e{*}y = y{*}e = y), \quad \forall y \exists z\ (x{*}y = y{*}z = e).$$

A group $\mathcal{M} = (G, \cdot, 1, =)$ is a model of the axioms for group theory: when we interpret $*$ as the multiplication operation $\cdot$ of G, and e as the neutral element 1 of G, the three formal expressions above are naturally recognized as assertions stating that the operation $\cdot$ and the element 1 satisfy the laws making $(G, \cdot, 1, =)$ a group.

Clearly the group axioms can have non-isomorphic models: for example $(\mathbb{Z}, +, 0, =)$ and $(\mathbb{Z}_n, +, [0]_n, =)$. Letting Γ be the group axioms and φ the formula

$$\forall x(\underbrace{x * \cdots * x}_{n-\text{times}} = e),$$

we have that $(\mathbb{Z}, +, 0, =)$ is not a model of φ, while $(\mathbb{Z}_n, +, [0]_n, =)$ is a model of φ, even though both satisfy the axioms of Γ.

Indeed this is the paradigm of independence. It is enough to show that there are two models of the same set of axioms Γ, one verifying a sentence φ, the other verifying its negation, in order to conclude that φ is *formally independent* from Γ.

In our case we are interested in structures for set theory, i.e. of the form $(M, E, =)$ with $E \subseteq M^2$ a binary relation which is the intended meaning for M of the $\in$-relation, and $=$ the equality relation on M. In this setting it is correct to state that $(V, \in, =) \models \mathsf{ZFC}$.

The discovery that the same axioms could have different models helped to develop an abstract perspective, shaping ideas and methods of modern mathematics. During the XIX century it was first discovered that even in the case of theories—i.e. collections of mathematical propositions—meant to describe a unique reality, like geometry, there might be different incompatible models. Concretely, this occurred with the discovery of models for geometry not satisfying Euclid's Parallel Axiom. This axiom is indeed independent from the other axioms of geometry, since it holds in the standard three, or two, dimensional euclidean space, but fails on the hyperbolic space or on the sphere.

This is exactly what Gödel's and Cohen's results achieve with respect to CH and ZFC. They provide two structures that satisfy all ZFC-axioms one verifying CH, the other falsifying it.

3.2. **Gödel's constructible universe.** The key idea of Gödel's construction consists in carving inside the universe V the minimal model of ZFC containing all ordinals Ord, and closed under the most basic set-theoretical operations, which Gödel reduced to the following list:

$$
\begin{array}{ll}
G_1(X,Y) = \{X, Y\} & G_2(X,Y) = X \times Y \\
G_3(X,Y) = \{(x,y) : x \in X, y \in Y, x \in y\} & G_4(X,Y) = X \setminus Y \\
G_5(X,Y) = X \cap Y & G_6(X) = \bigcup X = \{z : \exists y \in X\, z \in y\} \\
G_7(X) = \{x : (x,y) \in X\} & G_8(X) = \{(x,y) : (y,x) \in X\} \\
G_9(X) = \{(x,y,z) : (x,z,y) \in X\} & G_{10}(X) = \{(x,y,z) : (y,z,x) \in X\}
\end{array}
$$

The resulting structure was named the *constructible hierarchy* L by Gödel himself, given its extremely simple definition of a contructivist flavour. Moreover, Gödel showed that L satisfies ZFC and CH at the same time. Further investigations starting with the seminal work of Jensen [29] showed that L has such a simple structure that it is most often possible to compute, according to L, the solution (in L) of many problems undecidable (or open) on the basis of ZFC alone. We will detail more on this point later on.

It is possible to show that L is the smallest transitive model of ZFC, being the intersection of all transitive classes M contained in V, such that $(M, \in, =) \models \mathsf{ZFC}$ and M contain all ordinals. Roughly stated, L throws away many sets of V and retains just the minimal amount of sets needed to validate ZFC.

3.3. **Forcing.** The strategy adopted by Cohen to construct a model of ZFC where CH fails consisted in devising a completely new technique, named *forcing.*

While Gödel's constructible universe carves inside V the minimal model of ZFC, Cohen's forcing method takes an opposite approach and aims to enlarge V, adding sets that are "new" in a sense to be specified later, and thus producing a larger universe of sets. What is true or false in such a larger universe of sets depends on which "new" sets are added.

We cannot refrain from giving a brief description of some of the aspects of forcing. We have to admit that this is far from easy for both conceptual and technical reasons. There are several obstructions, one being that most mathematicians may complete their graduate studies without ever encountering a course in logic during their master or bachelor program. Hence we will try to give the flavour of Cohen's forcing method outlining analogies between forcing and other more familiar constructions.

The simplest analogy compares forcing with the adjunction of a polynomial root to a field. Moving from the structure $(\mathbb{Q}, \cdot, +, =)$ to the structure $(\mathbb{Q}(\sqrt{2}), \cdot, +, =)$ (which are both models of the field axioms), we find in the larger field both solutions of the equation $x^2 - 2 = 0$, which has no roots in $\mathbb{Q}$. The positive solution belongs to the set $\mathbb{Q} \cup \{\sqrt{2}\}$, but the latter set is not a field. Hence it is natural to enlarge it to $\mathbb{Q}(\sqrt{2})$, adding the least possible family of elements X such that $\mathbb{Q} \cup X$ is a field containing $\sqrt{2}$.

Much in the same way, Cohen's idea is to start from a transitive class (or set) M, such that $(M, \in, =) \models \mathsf{ZFC}$, and add a new set G to M satisfying certain desirable properties, so to build the smallest transitive model $(M[G], \in, =)$ of ZFC properly containing $M \cup \{G\}$.

But which models of set theory can we enlarge? Of course we cannot hope to enlarge V, the collection of all sets, because any G will be, itself, a set and therefore in V, hence we would have $V \subseteq V[G] \subseteq V$, ending up with what we started.

A way out to this logical difficulty is to perform a *boolean valued construction*[13]: one defines a new proper class $V^{\mathsf{B}}/_G$ and a new binary relation $\in^{\mathsf{B}} /_G$ on $V^{\mathsf{B}}/_G$ which is also a proper class different from the binary relation $\in$, thus showing the non-standard character of the model $(V^{\mathsf{B}}/_G, \in^{\mathsf{B}} /_G, =)$. These[14] two (proper) classes of sets are given by a well-defined property, and $(V^{\mathsf{B}}/_G, \in^{\mathsf{B}} /_G, =)$ is a structure where to test the truth of ZFC exactly the same way as $(V, \in, =)$ is. It turns out that $(V^{\mathsf{B}}/_G, \in^{\mathsf{B}} /_G, =)$ is a model of the ZFC-axioms.

---

[13]We roughly present this method following the alternative approach to forcing (with respect to Cohen's treatment) devised originally by Vopenka, Scott and Solovay [3].

[14]For the averted reader already familiar with forcing, we are here describing a scenario in which G is an ordinary ultrafilter on B, not a V-generic one, see [3] or the notes [26, 59] for more details on this approach.

We can also define a map $k : V \to V^{\mathsf{B}}/_G$ (of course, different from the identity) which embeds V as a substructure of $V^{\mathsf{B}}/_G$, i.e. is such that $a \in b$ if and only if $k(a) \in^{\mathsf{B}} /_G \, k(b)$. In this way we can view $V^{\mathsf{B}}/_G$ as an extension of (the image under k of) V.

Let us briefly sketch some considerations which guide the boolean valued construction[15]. We will borrow ideas from functional analysis and present a toy example of the forcing method which outlines that certain well known spaces of functions can also be seen as extensions of the real numbers on them obtained by means of forcing[16]. In particular our aim is to endow a certain ring of germs of measurable functions with a topological–algebraic–order–etc. structure resembling the natural one we have on the real numbers.

Suppose we want to describe a *new* real number. This real number should be *generic*, i.e. share any property which almost all real numbers have. For example it should be non-algebraic, since so are almost all real numbers, and it should be different from any fixed real number a, given that so are all other real numbers except a. Clearly such a new, generic real number cannot exist. Indeed, if it were some $b \in \mathbb{R}$, in order to be generic it should be different from itself, which is impossible.

To overcome this logical difficulty, the forcing method resorts to two clever ideas. First we might change the truth values we use, from $\{\text{True}, \text{False}\}$, the trivial boolean algebra, to a larger set, whose elements form a more complicated boolean algebra[17].

In the example below we use the complete boolean algebra MALG given by equivalence classes of Lebesgue measurable sets $A \subseteq \mathbb{R}$ modulo null sets[18]. Denoting by $[A]$ the equivalence class modulo null sets of the set A, the family of equivalence classes is a boolean algebra with operations $[A] \wedge [B] = [A \cap B]$, $\neg[A] = [\mathbb{R} \setminus A]$, $[A] \vee [B] = [A \cup B]$. It is a theorem (not always well known) that this boolean algebra is complete (i.e. admits suprema for all its subsets) with $\bigvee_{i \in I} [A_i] = [\bigcup_{i \in I} A_i]$.

To each sentence ϕ we can attach a boolean value $[\![\phi]\!] \in \mathsf{MALG}$ representing its truth-value. $[\mathbb{R}]$ is the truth, $[\emptyset]$ is the falsity, for any set A with A and $\mathbb{R} \setminus A$ both of positive Lebesgue measure $[A]$ denotes an intermediate truth value neither completely true nor completely false. The attribution of boolean values to properties ϕ should respect the logical structure of ϕ; for example a true sentence gets value $[\mathbb{R}]$, a false one gets value $[\emptyset]$, $[\![\phi \vee \psi]\!]$

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[15] We warn again the reader that the remainder of this section is far more advanced then the subsequent parts of the paper and that she/he can safely skip this part without compromising the comprehension of the following sections. For the remainder of this section we assume the reader is familiar with the basic properties of Lebesgue measure on $\mathbb{R}^n$ and of the space $L^\infty(\mathbb{R})$ given by essentially bounded measurable functions.

[16] A detailed account on what is sketched below can be found in [55].

[17] We refer the reader to [21] for the basic theory of boolean algebras.

[18] MALG stands for *Measure ALGebra*.

gets boolean value $[\![\phi]\!] \vee [\![\psi]\!]$ ($\phi \vee \psi$ denotes the disjunction of ϕ with ψ), $\neg\phi$ (the negation of ϕ) gets boolean value $\neg[\![\phi]\!]$, etc.[19]

In particular MALG gives us the means to properly interpret logical properties without committing ourselves to assert that a certain property is either true or false: there will be cases (see below) in which a certain logical property ϕ will get an intermediate value $[\![\phi]\!]$ neither true nor false.

The second clever idea of forcing regards the procedure to define the new elements. Consider the space of real-valued measurable functions. The elements of this space will be used to name the new real numbers according to a certain forcing construction. Indeed, we can identify $\mathbb{R}$ inside this space by means of the constant functions $c_a : x \mapsto a$ for each $a \in \mathbb{R}$.

Take now $\sin(x)$ and $\cos(x)$. We want to be able to decide whether these functions denote new, generic real numbers different from any $a \in \mathbb{R}$. If so, we want also to be able to decide which of the two denotes the bigger "new" real number. Forcing attaches to each Lebesgue measurable property $P(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ and to measurable functions $f_1, \ldots, f_n$ the equivalence class in MALG of the set $\{x \in \mathbb{R} : P(f_1(x), \ldots, f_n(x))\}$ [20]. For example we assign to the formula $\sin(x) = \cos(x)$ its boolean value $[\![\sin(x) = \cos(x)]\!] = [\{x \in \mathbb{R} : \sin(x) = \cos(x)\}]$ in MALG; since this is the equivalence class of a measure 0-set, generically $\sin(x)$ and $\cos(x)$ will name different reals. Similarly for each $a \in \mathbb{R}$

$$\{x \in \mathbb{R} : \sin(x) = c_a(x) = a\}$$

is countable, hence of measure 0. This means that $\sin(x)$ denotes a real number different from any $a \in \mathbb{R}$. Hence, a "new" real. On the other hand $\sin(x) < \cos(x)$ gets boolean value

$$[\{x \in \mathbb{R} : \sin(x) < \cos(x)\}] = [\bigcup_{n \in \mathbb{Z}} ((2n-1) \cdot \pi + \frac{\pi}{4}; 2n \cdot \pi + \frac{\pi}{4})]$$

and $\sin(x) > \cos(x)$ gets boolean value $[\bigcup_{n \in \mathbb{Z}} (2n \cdot \pi + \frac{\pi}{4}; (2n+1) \cdot \pi + \frac{\pi}{4})]$. Hence

$$[\![\sin(x) \neq \cos(x)]\!] = [\mathbb{R} \setminus (\pi \cdot \mathbb{Z} + \pi/4)] = [\![\sin(x) < \cos(x)]\!] \vee [\![\sin(x) > \cos(x)]\!],$$

and therefore it is true that $\sin(x)$ and $\cos(x)$ denote two different reals and that they are comparable, since to these sentences we assign the equivalence class of $\mathbb{R}$ in MALG.

On the other hand in order to decide whether $\sin(x) > \cos(x)$ holds or not, we must decide whether to choose $[\![\sin(x) < \cos(x)]\!]$ or $[\![\sin(x) > \cos(x)]\!]$, in particular the logical properties $\sin(x) < \cos(x)$ and $\sin(x) > \cos(x)$ gets both positive (but complementary) boolean values in MALG. Similarly we will have to decide whether the map $\arctan(x)$ is smaller or larger than $\sin(x)$ and/or $\cos(x)$, and such decisions must be made for all measurable functions.

[19] For those familiar with first order logic $[\![\exists x \phi]\!] = \bigvee_a [\![\phi(a)]\!]$ as a ranges in the appropriate domain. To interpret quantifiers we use that MALG is complete.

[20] Notice that if $P \subseteq \mathbb{R}^n$ is Lebesgue measurable, so is $\{x \in \mathbb{R} : P(f_1(x), \ldots, f_n(x))\}$.

Also, the decisions must be coherent, in the sense that it cannot be the case that we can choose simultaneously the boolean values $[\![\sin(x) < \cos(x)]\!]$, $[\![\cos(x) < \arctan(x)]\!]$, $[\![\arctan(x) < \sin(x)]\!]$, otherwise we would not end up having a linear order on these functions (which is a property we should have, if our family of new and old real numbers describes a new structure which satisfies most of the properties the continuum has).

A coherent selection of choices is possible resorting to the notion of ultrafilter G on MALG: a subset of MALG closed under $\wedge$ i.e. $[A], [B] \in G$ entails that $[A \cap B] \in G$, and such that exactly one among $[A]$ or $[\mathbb{R} \setminus A] \in G$ for any $[A] \in \mathsf{MALG}$; these conditions entail that $[\mathbb{R}] \in G$ and $[\emptyset] \notin G$, and also that whenever $[A] \in G$ and $B \supseteq A$, $[B] \in G$ as well. Roughly an ultrafilter G on MALG decides which among the $[A]$ belonging to the boolean algebra MALG are considered "true" (those in G), and the selection is coherent: i.e., if $[A]$ and $[B]$ are "true" so is $[A] \wedge [B]$; maximal: i.e., it always decide whether $[A]$ or $\neg[A]$ is "true"; and consistent with what MALG has already decided about truth: i.e., $[\mathbb{R}] \in G$ and $[\emptyset] \notin G$, and if $[A] \in G$ and $[B]$ is "more true" than $[A]$ according to MALG, i.e. $B \supseteq A$, then $[B] \in G$ as well [21] .

Now given G ultrafilter on MALG and a measurable function f, define $[f]_G = \{g : [\![f = g]\!] \in G\}$. Since just one among $[\![\sin(x) < \cos(x)]\!] \in G$ or $[\![\sin(x) \geq \cos(x)]\!] \in G$, G selects which of the two holds; on the other hand since $[\![\sin(x) = \cos(x)]\!] = [\emptyset]$, either $[\![\sin(x) < \cos(x)]\!] \in G$ or $[\![\sin(x) > \cos(x)]\!] \in G$. More generally we obtain that the relation $<_G$ given by $[f]_G <_G [g]_G$ if and only if $[\![f < g]\!] \in G$ defines a dense linear order with no end-points on $\{[f]_G : f \text{ is real-valued measurable}\}$.

Let us sketch a proof of the density property of $<_G$: Assume $[f]_G <_G [g]_G$, we must find some $[h]_G$ such that $[f]_G <_G [h]_G <_G [g]_G$. Now $[f]_G <_G [g]_G$ if and only if $[\![f < g]\!] = [\{x \in \mathbb{R} : f(x) < g(x)\}] \in G$. Let $h(x) = \frac{f(x)+g(x)}{2}$. Then

$$[\![f < h]\!] \wedge [\![h < g]\!] = [\![f < h < g]\!] = [\{x \in \mathbb{R} : f(x) < h(x) < g(x)\}]$$

Since

$$\{x \in \mathbb{R} : f(x) < h(x) < g(x)\} = \{x \in \mathbb{R} : f(x) < g(x)\},$$

and $[\{x \in \mathbb{R} : f(x) < g(x)\}] \in G$, we get that $[\![f < h]\!], [\![h < g]\!]$ are both in G, yielding that $[f]_G <_G [h]_G <_G [g]_G$.

[21] For the specific case we are considering we can give an alternative description of the notion of ultrafilter. Consider the real C^*-algebra $L^\infty(\mathbb{R})$ given by real-valued measurable functions which are also essentially bounded. A character $\theta : L^\infty(\mathbb{R}) \to \mathbb{R}$, is a continuous homomorphism of this real C^*-algebra onto $\mathbb{R}$. Denote by χ_A the characteristic function of a measurable set A, and given a character θ let $G_\theta = \{[A] : \theta(\chi_A) = 1\}$. Then G_θ is an ultrafilter on MALG. Conversely given a ultrafilter G on MALG define a character $\theta_G : L^\infty(\mathbb{R}) \to \mathbb{R}$ letting $\theta_G(\sum_{i \in I} \lambda_i \chi_{A_i}) = \sum_{[A_i] \in G} \lambda_i$ whenever $\{A_i : i \in I\}$ is a partition of $\mathbb{R}$ in Lebesgue measurable sets; we can use the density of linear combinations of characteristic functions of measurable sets in $L^\infty(\mathbb{R})$ to extend θ_G to all of $L^\infty(\mathbb{R})$. In particular the two notions of character and ultrafilter are completely equivalent for MALG.

We have just outlined a very special case of a deeper result. Consider the space $L^{\infty+}(\mathbb{R})$ given by measurable functions $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty, -\infty\}$ such that $\{a \in \mathbb{R} : f(a) = \pm\infty\}$ has Lebesgue measure 0 [22]. Given a ultrafilter G on MALG, define the ring of germs $L^{\infty+}(\mathbb{R})/_G$ letting $[f]_G = [g]_G$ if and only if $A_{f,g} = [\{x : f(x) = g(x)\}]$ is in the ultrafilter G on MALG. It is actually possible to check that $L^{\infty+}(\mathbb{R})/_G$ with its pointwise operations modulo G (i.e. $[f]_G \cdot_G [g]_G = [f \cdot g]_G$, etc.) is a field [23].

The forcing theorem, which is the key to understand what is true in $L^{\infty+}(\mathbb{R})/_G$, can be phrased as follows in this specific context:

Given a Borel[24] *relation $R \subseteq \mathbb{R}^n$, a ultrafilter G on* MALG*, $f_1 \dots, f_n \in L^{\infty+}(\mathbb{R})$, we say that*

$\bar{R}/_G([f_1]_G, \dots, [f_n]_G)$ *holds if and only if* $[\{a \in \mathbb{R} : R(f_1(a), \dots, f_n(a))\}] \in G$.

Then for R, S Borel relations on $\mathbb{R}^n$, their lifts $\bar{R}/_G$, $\bar{S}/_G$ behave properly, for example:

- $\overline{\mathbb{R}^n \setminus R}/_G([f_1]_G, \dots, [f_n]_G)$ *holds if and only if* $\bar{R}/_G([f_1]_G, \dots, [f_n]_G)$ *does not,*
- $\overline{R \cap S}/_G([f_1], \dots, [f_n])$ *if and only if* $\bar{R}/_G([f_1]_G, \dots, [f_n]_G)$ *and* $\bar{S}/_G([f_1]_G, \dots, [f_n]_G)$*,*
- *the same conclusion holds for the (possibly non Lebesgue measurable) relation on $\mathbb{R}^{n-1}$ obtained by projecting along an axis a Borel relation on $\mathbb{R}^n$, and for the Borel relation on $\mathbb{R}^n$ given by the countable union (or intersection) of Borel relations $R_j \subseteq \mathbb{R}^n$ for $j \in \mathbb{N}$.*

More generally given an n-ary Borel relation $R \subseteq \mathbb{R}^n$ and a first order formula $\phi(x_1, \dots, x_n)$ in a language with one n-ary relation symbol

$$(L^{\infty+}(\mathbb{R})/_G, \bar{R}/_G, =) \models \phi([f_1]_G, \dots, [f_n]_G)$$

if and only if

$$[\{a \in \mathbb{R} : (\mathbb{R}, R, =) \models \phi(f_1(a), \dots, f_n(a))\}] \in G.$$

This theorem is extremely powerful, for it links the logical properties which hold in $L^{\infty+}(\mathbb{R})/_G$ to the combinatorial properties of the boolean algebra MALG and of the ultrafilter G.

We are just scratching the surface of the enormous complications one has to overcome to deal properly with forcing. The general method of forcing

[22] For example the identity, $x \mapsto 1/x$, the exponential map are examples of measurable functions in $L^{\infty+}(\mathbb{R}) \setminus L^{\infty}(\mathbb{R})$, i.e. measurable functions which are not essentially bounded. Strictly speaking $x \mapsto 1/x$ is not even real-valued measurable since it is not defined on all of $\mathbb{R}$ but just on a conull subset of $\mathbb{R}$, this is one of the reason to extend the possible values of the functions to $\mathbb{R} \cup \{+\infty, -\infty\}$, while restricting the new set of values to have measure 0.

[23] Here we are really using that the space we work with is $L^{\infty+}(\mathbb{R})$, for example for example the inverse of $[\sin(x)]_G$ is the equivalence class of the function $\frac{1}{\sin(x)}$ which belongs to $L^{\infty+}(\mathbb{R}) \setminus L^{\infty}(\mathbb{R})$, while the measurable function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty, -\infty\}$ with constant value $+\infty$ is such that $[f]_G$ does not have an inverse.

[24] See the beginning of section 5.2 for the definition of Borel set. Remark that Borel sets are Lebesgue measurable.

applied to the structure $(V, \in, =)$ is able to encode the above situation as follows: the reals are the real numbers of V, and one is able, by means of forcing, to construct a structure $V^{\mathsf{MALG}}/_G$ such that the real numbers of $V^{\mathsf{MALG}}/_G$ are exactly the elements of the ring $L^{\infty+}(\mathbb{R})/_G$. In particular in $V^{\mathsf{MALG}}/_G$ there are new real numbers, for example $[\sin(x)]_G$, with respect to those existing in V.

However to define properly $V^{\mathsf{MALG}}/_G$ we should: **(a)** besides describing how to define the new real numbers in the larger model $V^{\mathsf{MALG}}/_G$, define what counts as a natural number, a complex number, etc. in $V^{\mathsf{MALG}}/_G$. More generally one must be able to define all set-theoretical operations in $V^{\mathsf{MALG}}/_G$, since it must be shown that $(V^{\mathsf{MALG}}/_G, \in^{\mathsf{MALG}}/_G, =)$ models ZFC; **(b)** be able to define the forcing theorem in general for models of the form $V^{\mathsf{B}}/_G$ where B is a complete boolean algebra and G is a ultrafilter on it; **(c)** finally, in order to use forcing to establish the independence of a statement φ (such as the continuum hypothesis), we should find some B so that we can compute whether $V^{\mathsf{B}}/_G$ models φ or not.

The very ingenious strategy of Cohen was to use the forcing theorem to transfer the problem of checking whether CH holds or not in $V^{\mathsf{B}}/_G$ to the problem of checking whether B satisfies certain combinatorial properties. He was then able to prove that the algebra C given by the regular open sets of the space $[0,1]^{\mathcal{P}(\mathcal{P}(\mathbb{N}))}$ endowed with the product of the euclidean topology on $[0,1]$ is such that $V^{\mathsf{C}}/_G$ models[25] $\neg\mathsf{CH}$.

4. Gödel's program

The set theoretical tools developed by Gödel and Cohen show that the phenomenon of independence in mathematics is broader than expected and not confined to *ad hoc* examples, as those discovered by Gödel, in 1931, by means of his incompleteness theorem. Indeed CH is a mathematical problem which grew out of the researches of the most prominent mathematicians of the end of the XIX century. The combined use of Gödel's constructible hierarchy and Cohen's forcing method showed the independence of a variety of mathematical problems arising in distinct fields, such as group theory, e.g. the Whitehead problem on the characterization of free groups [49], [50], functional analysis, e.g. Kaplansky characterization of Banach algebra morphisms [11], or the problem of the existence of outer automorphisms for the Calkin algebra [12, 47], and many, many others. These results showed the inadequacy of ZFC to give a complete and satisfactory picture of a fast growing discipline like contemporary mathematics. There are "outskirts" of the mathematical universe where the ordinary methods of proof do not

[25] For a complete presentation of this method see [3, 34] or the notes [59]. Remark that (modulo the identification of Cohen's original poset — given by partial functions $p : \omega_2 \times \omega \to 2$ with finite domain — with its boolean completion, and the assumption of Cohen that $\aleph_2 = |\mathcal{P}(\mathcal{P}(\mathbb{N}))|$ holds in the ground model) the above example is exactly the original forcing notion devised by Cohen.

suffice to find an answer to well-posed and natural problems, at least this is not possible with the means offered by ZFC. Nonetheless set theoretic techniques are crucial to understand which problems inhabit this hazy part of mathematics.

4.1. **How to overcome the weakness of ZFC?** Some scholars, like Cohen himself [10], or Feferman, argue that the independence results with respect to ZFC cast shadows on the possibility that a mathematical theory of infinity could be captured axiomatically. But independence with respect to ZFC, by itself, does not prove the existence of intrinsic limits of the axiomatic method, but only of a specific axiomatic system: namely ZFC. An approach offered by the set-theoretic community, and suggested by Gödel himself, consists in supplementing the ZFC-axioms with *new* set theoretical principles, able to give solutions to the largest possible family of otherwise independent, or open, problems. This approach goes under the name of *Gödel's program*.

The practice of extending the means of proofs with new principles asserting the existence of certain mathematical entities is as old as mathematics itself, even if it often took time to incorporate these principles in the main body of mathematics. Think about the introduction of irrational numbers following the Pithagorean discovery of the irrationality of $\sqrt{2}$, or the use of complex numbers to find roots of third degree polynomial equations. As a more recent example, Groethendieck universes[26], mathematical objects whose existence cannot be proved in ZFC, were used in Wiles original solution to Fermat's last theorem. However, there is a difference between the use of new mathematical principles in Wiles proof and the extensions of ZFC originated by Gödel's program: their unequal *status of necessity*. As a matter of fact a solution of Fermat's last theorem which avoids any reference to Groethendieck universes and that uses elementary methods—i.e. formalizable in Peano's Arithmetic or at least in ZFC—is believed by many to be possible, although unknown, at least for the case of Peano's Arithmetic. On the other hand it is provably impossible to solve CH on the basis of ZFC. To give a solution to CH it is *necessary* to supplement ZFC with new axioms.

As already mentioned, there are scholars, like Feferman [13], who doubt the existence of, yet unknown, set-theoretic truths, able to decide CH. But in our opinion the situation is not much different than it was in other well-known turning points in the development of mathematics. One example is the debate surrounding the introduction of AC at the beginning of the XX century [43]. While AC brings some undesirable consequences, such as the existence of non-measurable sets of reals, it has been finally accepted

[26]Groethendieck universes are mathematical entities providing a correct framework where certain category theoretic questions can be properly formulated and solved. Their existence is not provable in ZFC but follows from the axiom stating the existence of a strongly inaccessible cardinal; see [51] for an account on Groethendieck universes, in section 5.1 we give the definition of strongly inaccessible cardinals.

by most mathematicians. A compelling reason being that by means of AC it is possible to give a simple general outline of many mathematical theories: one example is the nice general theory of cardinalities we sketched in 2.2; other fundamental consequences of AC are the Hahn-Banach theorem, the existence of prime ideals in rings, the compactness of the product of compact spaces, just to name few of them. Nowadays most textbooks in functional analysis, algebra, topology, etc. do not question the truth of AC and develop their field using it freely. Following Hilbert's motto "Wir müssen wissen – wir werden wissen"[27], we consider a feasible mathematical task to find the correct axioms for set theory which can settle CH, and two promising candidates are outlined in the final section of this paper. We are also confident that time and practice can bring the mathematical community to accept these axioms much in the same way it occurs now for AC. Of course the adoption of new axioms will require both mathematical and philosophical arguments, since not only these axioms should be able to solve old problems—a good mathematical point in favor of their acceptance— but they also need to be well *justified*—by philosophical arguments.

4.2. **Intrinsic vs. extrinsic.** There is a vast literature on the criteria for justification of new axioms in set theory[28]. To give a rough idea of the discussion, we recall two famous quotes by Gödel [23], where he introduced the two kinds of justification that still occupy a central role in this debate: *intrinsic* and *extrinsic* justifications[29]. Let us start with the intrinsic ones.

> For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of". [...] probably there exist others based on hitherto unknown principles; also there may exist, besides the ordinary axioms, the axioms of infinity and the axioms mentioned in footnote 17 [here Gödel means large cardinal axioms] other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts [23, pp. 260].

On the other hand, extrinsic justifications assimilate set-theoretical methodology to that of empirical sciences.

> Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic

[27] We must know – we will know.

[28] Besides the classical contribution by Gödel, discussed in this section, we advise the interested reader to consult the work of Boolos [5], Maddy [37, 38], Koellner [32], and a more comprehensive discussion in [14].

[29] However, there are also new perspectives on justification in set theory that try to overcome this classical dicothomy [2].

> necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success", that is, its fruitfulness in consequences and in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs [23, pp. 261].

Although these two forms of justification seem to offer very different criteria, it is a well balanced mixture of both that is commonly used to give reasons for the acceptance of new axioms.

5. Large cardinals, determinacy axioms, and generic absoluteness for second order arithmetic

5.1. **Large cardinals.** The simplest and first natural examples of new axioms extending ZFC are large cardinal axioms. These axioms formalize the idea that the process of generation of new levels of the cumulative hierarchy of V is never completed.

The first example of a large cardinal axiom is the axiom of infinity: it can be shown that $(V_\omega, \in, =)$ is a model of all other ZFC-axioms; but in V_ω there are no infinite sets. This shows that it is not enough to appeal to the construction and existence principles given by the other axioms of ZFC to assert the existence of an infinite set, since there is a model of all these axioms in which no infinite set exists, but that we really need to postulate the existence of infinite sets. Remark also that, viewed as a suborder, any cofinal subset of $\mathbb{N}$ has the same isomorphism type of the natural numbers, i.e. $(\omega, \in)$. Generalizing these two observations one can produce the simplest example of a proper large cardinal axiom: δ *is strongly inaccessible* if $(V_\delta, \in, =) \models$ ZFC and any cofinal subset of δ has order type δ. By Gödel's incompleteness theorem ZFC cannot prove the existence of strongly inaccessible cardinals, otherwise it would prove its own consistency. In particular a strongly inaccessible cardinal is an ordinal whose description cannot be obtained starting from some smaller ordinal and appealing just to the construction principles given by the ZFC-axioms.

Many other large cardinal axioms have been formulated in the past century and are still introduced nowadays. As Gödel anticipated in his writings [23], there are very good reasons to argue in favor of their intrinsic plausibility; for an overview of these arguments see also [33]. Nonetheless in this paper we decide not to pursue any further this type of arguments. The reason being that soft and/or detailed accounts on this topic are already available[30]. Moreover an exposition of the basic properties of the most important large cardinal axioms, and of the intrinsic reasons to accept them would require us to sketch far more set theory than what has been briefly outlined in §1 and §2[31]. We will instead focus on the extrinsic reasons to accept large cardinal axioms, outlining in the other parts of this section the striking consequences on the properties of real numbers one can draw from them. This is a topic far less explored in the literature, for which an introductory account as the one we sketch below is in our opininon –at least to some extent– still missing[32].

5.2. **Regularity properties for sets of reals and determinacy.**
Given a topological space [33] (Y, τ) and $X \subseteq Y$:

- X is Borel if it belongs to the smallest σ-algebra containing the open sets (a σ-algebra on $\mathcal{P}(Y)$ is a family of subsets of Y closed under countable intersections, countable unions, and complements).
- X is Lebesgue measurable (now noted as $\mathsf{LM}(X)$) if it belongs to the σ-algebra generated by the Lebesgue measure on $\mathbb{R}$.
- X has the property of Baire ($\mathsf{BP}(X)$), if there is an open set U such that $U \Delta X$ (the symmetric difference) is a meager set (i.e. the countable union of nowhere dense sets).
- X has the property of the perfect set ($\mathsf{PSP}(X)$), if it is either countable or has a nonempty perfect subset (which is a closed set with no isolated points). Cantor proved that any subset of $\mathbb{R}$ with the

[30]For a soft introduction to large cardinals see for example the nice introductory paper by Koellner available at *logic.harvard.edu*. The standard reference for large cardinals and their properties is Kanamori's [30].

[31]A warning to the reader is in order at this point: we will not define in this paper any other large cardinal notion. Nonetheless at several points of the discussion to follow we will mention a variety of large cardinals, including (in order of increasing strength): strongly inaccessible cardinals, measurable cardinals, strong cardinals, Woodin cardinals, supercompact cardinals, etc. We refer the reader to Koellner's paper or to Kanamori's book for details on the definitions and the properties of these cardinals. Let us just mention that a crucial and surprising feature of these axioms is that they line up in a well-ordered linear hierarchy, for example: supercompacts are Woodin; if δ is Woodin, it is strongly inaccessible and $(V_\delta, \in, =)$ models that there are strong cardinals; a strong cardinal is measurable; a measurable cardinal is strongly inaccessible. A chart describing the known dependencies between the most important large cardinal axioms can be found in the last pages of Kanamori's book.

In the remainder of this paper we will outline certain consequences these large cardinals have on the universe of sets. These consequences are either mathematical facts which refer to familiar mathematical concepts, or mathematical facts which can be meaningfully formulated on the basis of the set theory sketched so far. In particular there is no need to define any of the above large cardinal axioms in order to outline the consequences we will draw from them, and we will not do that here.

[32]See however the beautiful papers by Koellner [32] or by Woodin [63].

[33]To grasp the basic ideas guiding the definitions to follow the reader may assume that Y is the set of real numbers with the usual topology.

perfect set property is either countable or in bijection with $\mathbb{R}$, hence cannot be a counterexample to[34] CH.

Already at the beginning of the XX century it was known, using AC, how to build sets without the Baire property (Bernstein), or without the perfect set property (Bernstein, again), or non-Lebesgue measurable (Vitali). On the other hand the combined works of Cantor, Bendixson, Alexandrov, Lebesgue, Borel, and others showed that all Borel sets have all the above regularity properties. In 1970 Solovay [52] proved that if there is a strongly inaccessible cardinal—one of the lowest in the hierarchy of large cardinals—it is consistent that all subsets of the reals are Lebesgue measurable, have the property of Baire, and have the perfect set property. Stated more precisely: it is possible to construct a model of set theory in which choice fails (i.e. a model of the axiom system $\mathsf{ZFC} \setminus \{\mathsf{AC}\}$) and all the above properties hold; moreover Solovay's model satisfies also a weak form of the Axiom of Choice, the Axiom of Dependent Choices[35] DC.

The study of regularity properties of subsets of $\mathbb{R}$ is greatly simplified by the observation that for Borel (or topologically more complex) sets it does not matter whether we consider them as subsets of $\mathbb{R}$ or of any other Polish space[36], such as the Cantor space $2^{\mathbb{N}}$ or the Baire space $\mathbb{N}^{\mathbb{N}}$. The reason being that all these spaces are *Borel isomorphic*, that is in bijection via a Borel map with a Borel inverse[37]. For this reason, without loss of generality, we can restrict our discussion to the space $2^{\mathbb{N}}$ endowed with the product topology.

The game $\mathcal{G}_A$ with players I and II and payoff the set $A \subseteq 2^{\mathbb{N}}$ is defined according to the following rules: it lasts ω-moves; the two players alternate their moves with I playing a_{2n} at even stages $2n$, II playing a_{2n+1} at odd stages $2n+1$; after ω-moves I wins if $\langle a_n : n \in \mathbb{N} \rangle \in A$, II wins otherwise.

A strategy for player I is a map $\sigma : 2^{<\mathbb{N}} \to 2$ which for any given partial play $\langle a_0, \ldots, a_{2n+1} \rangle$ tells player I what is the move to make in this case, i.e. $\sigma(\langle a_0, \ldots, a_{2n+1} \rangle)$. Hence a run of the game according to σ looks like

$$\langle a_0 = \sigma(\emptyset), a_1, a_2 = \sigma(\langle a_0, a_1 \rangle), a_3, a_4 = \sigma(\langle a_0, a_1, a_2, a_3 \rangle), a_5, \ldots \rangle$$

[34] The attempt to verify CH by showing the perfect set property for subsets of the reals of increasing topological complexity gave rise to a very interesting line of research, which is now part of *descriptive set theory* (see [31]).

[35] DC states that for all non-empty set X and every $\sigma : X^{<\mathbb{N}} \to X$, there exists $f : \mathbb{N} \to X$ such that $f(n) = \sigma(f \restriction n)$ for all $n \in \mathbb{N}$. It allows to construct infinite sequences in X obeying the constraints imposed by σ.

[36] A topological space (X, τ) is Polish if it is separable (i.e. it has a countable dense subset) and there is a distance d on X with the property that (X, d) is a complete metric space such that the balls $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ generated by d are a basis for the topology τ.

[37] $f : X \to Y$ is Borel if the preimage of Borel sets is Borel.

We say that σ is a winning strategy for I if any run of the game according to σ is won by I no matter what II plays[38] . Similarly one defines a winning strategy for player II. It is not hard to check that games of finite length with perfect information, like Chess or Noughts and Crosses, can be coded by games of type[39] $\mathcal{G}_A$, by deciding which of the two players wins in case of a draw. A result by Zermelo gives that all such finite games are *determined*, that is exactly one of the two players has a winning strategy[40].

In 1962 Mycielski and Steinhaus [46] introduced the Axiom of Determinacy AD, stating that $\mathsf{AD}(A)$ holds for all $A \subseteq 2^{\mathbb{N}}$, where $\mathsf{AD}(A)$ says that exactly one of the two players has a winning strategy, i.e. $\mathcal{G}_A$ is determined.

AD was proposed as an alternative to AC, being provably false assuming ZFC, and it fits with a fruitful line of research on games that was started during the 20s and the 30s by Mazur, Banach, and Ulam among others[41]. It is a result, already from the 50s, by Gale and Stewart [20], that games with a closed or open payoff set are determined. A major surprise came in the early seventies when Martin [39] proved that the existence of a measurable cardinal entails that all games with analytic[42] payoff set are determined. Moreover, Martin later showed [40], in ZFC, that games with a Borel payoff set are determined, this time avoiding any use of large cardinals. These subsequent results are normally considered as extrinsic reasons for the claim that large cardinals are well-justified principles extending ZFC. As we hinted before, we see here at play a model of justification that resembles one of prediction and confirmation from natural sciences.

The use of consequences of AD greatly simplifies the study of regularity properties. Indeed a huge variety of regularity properties, including the perfect set property, the Baire property, and Lebesgue measurability, follow from the determinacy of an appropriate game in combination with[43] DC.

A key definition, isolated by Feng, Magidor, and Woodin in [15] is that of universally Baire sets of reals:

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[38]It is not transparent that non-trivial runs of the games according to a strategy σ always exist. This can be proved appealing to the Axiom of Dependent Choices.

[39]In the definition of infinite games of length ω it is convenient to assume that a draw is not possible to simplify many arguments. This is not restrictive: optimal strategies for games in which the players can also get to a draw can be recovered by combining the winning strategies of the two games obtained by letting either of the two players win in case of a draw.

[40]Zermelo's proof is non constructive, for example in the case of chess we do not (as yet) know which of the two players has a winning strategy; once again assuming that a draw is a winning condition for one of the two players.

[41]See [36] for an overview of this history and [31] for some of the uses of such games in the study of topological properties of spaces of functions.

[42]$A \subseteq 2^{\mathbb{N}}$ is analytic if it is the continuous image of some Borel set. Clearly Borel sets are analytic, as witnessed by the identity function.

[43]One needs DC to grant that for certain sets B (depending on the regularity property one wants to establish for A) there always are infinite runs of the games $\mathcal{G}_B$ according to the winning strategy given by $\mathsf{AD}(B)$.

> Given a topological space X, $A \subseteq 2^{\mathbb{N}}$ is X-Baire if $f^{-1}[A]$ has the Baire property in X for all $f : X \to 2^{\mathbb{N}}$ continuous. Moreover we say that A is universally Baire if it is X-Baire for all compact Hausdorff topological spaces X.

Not only Borel subsets of $2^{\mathbb{N}}$ are universally Baire (this is provable in ZFC), but assuming the existence of a proper class of Woodin cardinals[44] so are all sets definable in second order arithmetic, i.e. those subsets of $2^{\mathbb{N}}$ definable with parameters in the first order structure $\langle \mathbb{N}, \mathcal{P}(\mathbb{N}), \in, 0, +, = \rangle$ or, equivalently, in the first order structure $(V_{\omega+1}, \in, =)$[45].

A major result of Martin and Steel [41], combined with the work of Woodin [35], establishes that if one assumes a proper class of Woodin cardinals then all universally Baire sets of reals are determined, i.e. $\mathcal{G}_A$ is determined if A is universally Baire. Therefore all nice regularity properties hold for universally Baire sets assuming large cardinals.

Summing up our discussion: AC has undesirable consequences, as subsets of $\mathbb{R}$ failing to have certain regularity properties, but it also gives the means to prove in full generality certain expected results which are essential for several fields of mathematics. Large cardinals imply that $\mathsf{AD}(A)$ holds for sets of reals A defined by sufficiently simple topological properties, i.e being universally Baire; this suffices to prove that all universally Baire sets A have all desirable regularity properties. Moreover AC and determinacy arguments are both widely used in various branches of mathematics. Finally AC and determinacy in its full strength contradict each other, but they coexist harmoniously provided one puts natural bounds, given by the notion of universal Baireness, on the topological complexity of the sets for which determinacy holds.

5.3. **Generic absoluteness for second order arithmetic.** In an unexpected turn of events a deep and fruitful connection between large cardinals and forcing was then discovered by Woodin [35, Thm 3.3.13, Section 3.4]. Informally this result, which goes under the name of *Generic absoluteness for second order arithmetic*, can be stated as follows.

> Assume there are class many Woodin cardinals. Given a mathematical problem formalizable in second order arithmetic, if a solution can be established by means of forcing, then that is the correct solution.

[44] The expression "a proper class of large cardinals with a given property ϕ" asserts that the family of large cardinals satisfying ϕ is a proper class. It is out of the scope of the present paper to define Woodin cardinals (the interested is referred to [53]), let us just remark that (at least in our opinion) if one is eager to accept the intrinsic reasons which justify the existence of inaccessible cardinals, then she should have no problems to accept the axioms asserting the existence of Woodin cardinals on the basis of the same "intrinsic" arguments.

[45] This family of sets can also be characterized topologically as the family of projective sets i.e. those obtained by a Borel subset of $(2^{\mathbb{N}})^n$ applying repeatedly either the operation of projection on some coordinate or the operation of complementation.

That is, if we accept large cardinal axioms, we can turn forcing into a useful tool for proving direct implications rather than independence results. Indeed, it is sufficient to show that forcing produces a structure where a sentence formalizable in second order arithmetic holds, in order to turn this consistency proof into a direct derivation of the given sentence from large cardinal assumptions. In particular Woodin's result rules out the possibility to use forcing to prove the independence, from ZFC supplemented with large cardinals, of any problem formalizable in second order arithmetic.

Finally a vast portion of number theory, analysis, differential geometry, measure theory, and probability theory can be expressed by sentences in second-order arithmetics, hence for a great number of problems arising in these fields, we might try to use Woodin's result directly and solve such problems establishing the consistency of their solution by means of forcing; this approach is taken for example in [6, 61].

6. Axioms able to settle CH.

There are many important ZFC-undecidable problems which cannot be properly formalized in second order arithmetic. Woodin's results cannot apply to these problems. Among these we mention once again CH, Whitehead's problem on free groups, the existence or not of outer automorphism of the Calkin algebra. This motivates the two programs we are about to discuss. Both propose to adopt stronger axioms able to give a unified picture of a larger portion of the universe of sets; a portion large enough to include almost all sets definable in third order arithmetic (i.e. expressible in the first order structure $(V_{\omega+2}, \in, =)$). As a matter of fact, a unified theory of this portion of the universe would give a solution to all problems mentioned above.

The first program consists in a step by step strategy aimed at giving a complete picture of larger and larger initial segments of the cumulative hierarchy, with the goal of finding an analogue of Woodin's absoluteness result, first, for third order arithmetic, and then for more and more complex initial segments of V. The second program tries to find global properties able to give a general, detailed, picture of the whole universe of sets V, once and for all.

6.1. Forcing axioms. Intuitively, forcing axioms tell us that the universe of all sets has been saturated by means of the possibilities given by the method of forcing. As a consequence we could think of a structure satisfying forcing axioms as one obtained after many applications of forcing. Indeed this is the rough idea behind their relative consistency proofs.

Mathematically, forcing axioms can be presented in several different ways. The most common is to view them as generalizations of the Baire category theorem BCT. Recall that BCT states that for any (locally) compact Hausdorff space X the intersection of countably many open dense subsets of X is still dense, and therefore non empty. On the other hand it is not hard to find uncountable collections of dense open subsets of $\mathbb{R}$ with empty intersection, for example

$$\bigcap_{x\in\mathbb{R}}(\mathbb{R}\setminus\{x\})=\emptyset.$$

A more sophisticated example is the following. Consider the one point compactification $X=\aleph_1\cup\{*\}$ of the space $\aleph_1$, endowed with the discrete topology. Take the compact Hausdorff space $X^{\mathbb{N}}$ with the product topology. The sets $E_\alpha=\{f\in X^{\mathbb{N}}:\ \exists n\, f(n)=\alpha\}$ are open dense in $X^{\mathbb{N}}$, for all $\alpha\in X$, but $\bigcap_{\alpha<\omega_1}E_\alpha$ is empty. Indeed any g belonging to this intersection would be a surjection of the countable set $\mathbb{N}$ onto the uncountable set $\aleph_1\cup\{*\}$, but such a g clearly cannot exist.

Forcing axioms can be defined as suitable strengthenings of BCT:

> Given a cardinal λ and a topological space X, $\mathsf{FA}_\lambda(X)$ holds[46] if any family of λ-many dense open subsets of X has a non-empty intersection.

Therefore BCT is the statement that $\mathsf{FA}_{\aleph_0}(X)$ holds for all locally compact Hausdorff spaces X. On the other hand the above examples show that $\mathsf{FA}_{\aleph_1}(X)$ must fail for some compact space X. Nonetheless one of the driving questions which led the research in set theory during the past decades has been to isolate the largest class of compact Hausdorff spaces X for which $\mathsf{FA}_{\aleph_1}(X)$ can possibly hold. This would offer a forcing axiom able to settle a vast number of mathematical problems at once. Indeed, a long list of striking independence results were proved by showing that one solution to the problem holds in L, while its negation can be proved using the fact that $\mathsf{FA}_{\aleph_1}(X)$ holds for certain compact spaces X. More specifically, using forcing, it was possible to produce a model of ZFC where $\mathsf{FA}_{\aleph_1}(X)$ was true for the compact spaces X in question. For example this has been a successful strategy to prove the independence of all the problems mentioned at the beginning of this section.

Shelah, Magidor and Foreman [19] isolated a property of compact Hausdorff spaces X, that of being[47] *stationary set preserving* (noted $\mathsf{SSP}(X)$), which is provably in ZFC a necessary condition in order for $\mathsf{FA}_{\aleph_1}(X)$ to hold; but they were also able to show that this can also be a sufficient condition. Indeed, if a supercompact cardinal exists, then there is a model of ZFC such that the following holds:

(MM) For X compact Hausdorff, $\mathsf{FA}_{\aleph_1}(X)$ if and only if $\mathsf{SSP}(X)$.

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[46] FA_λ stands for *Forcing Axiom for λ-sized* families of dense open sets.

[47] $C\subseteq\omega_1$ is a club if it contains the supremum of all its countable subsets; $S\subseteq\omega_1$ is stationary if it meets all the club subsets of ω_1. $\mathsf{SSP}(X)$ holds if letting B be the complete booolean algebra given by regular open subsets of X and S be a stationary subset of ω_1, the forcing notion associated to B preserves the stationarity of S, i.e. — following the boolean valued approach to forcing given (for example) in [59]— we have that $[\![\check{S}\text{ is stationary }]\!]_{\mathsf{B}}=1_{\mathsf{B}}$.

This principle is known in the literature as *Martin's Maximum*, and predicates that $\mathsf{FA}_{\aleph_1}(X)$ holds for the largest possible family of locally compact spaces X, giving a maximal topological strengthening of BCT.

We can also argue that forcing axioms can also be presented as natural strengthenings of AC:

> (X, τ) is $< \lambda$-closed if whenever $\{A_i : i \in I\} \subseteq \tau$ is a family of size less than λ of non-empty open sets linearly ordered by inclusion, then $\bigcap_{i\in I} A_i$ contains a non-empty open set.

Notice that $\mathbb{R}$ with the euclidean topology is not $< \aleph_1$-closed. On the other hand this is the case for the space $2^{\aleph_1}$ with bounded topology, which is the topology generated by taking as a base the sets $N_s = \{f \in 2^{\aleph_1} : f \supseteq s\}$ as s ranges over the functions in 2^{α}, and α among the ordinals in $\aleph_1$.

Let Γ_λ denote the class of compact Hausdorff spaces which are $< \lambda$-closed. Goldblatt in [24] noted that AC is equivalent to the assertion that

(1) For all cardinals λ $\mathsf{FA}_\lambda(X)$ holds for all X in Γ_λ.

It is possible to check that $< \aleph_1$-closed Hausdorff compact spaces are stationary set preserving and it is immediate to check that all compact Hausdorff spaces are $< \aleph_0$-closed. With this in mind, BCT is the weakening of AC obtained by requiring (1) to hold just in case $\lambda = \aleph_0$, while MM is an optimal strenghtening of (1) for $\lambda = \aleph_1$.

Finally let us mention that there are also generic absoluteness results for third order arithmetic which follow from forcing axioms, as well as nice model theoretic properties for the models of these axioms. In a very precise sense, there are natural strenghtenings of Martin's Maximum, which are consistent relative to large cardinal axioms, and produce generic absoluteness results for third order arithmetic. This is a surprising and strong analogy with Woodin's result that large cardinals give generic absoluteness for second order arithmetic [62, 60, 1]. Moreover it is possible to give a very nice picture, inspired by model theoretic arguments, of the family of first order models of these axioms [58].

These absoluteness results for third order arithmetic give a logical explanation of the success forcing axioms have met in solving problems of that complexity. Indeed, MM implies that $|\mathbb{R}| = \aleph_2$, decides negatively Whitehead problems, and forces all automorphism of the Calkin algebra to be inner. The list of problems which are independent with respect to ZFC, and are solved assuming MM, is long and stretches from general topology, to functional analysis, algebra, and group theory; a non-exhaustive sample can be found in [44, 45]. Moreover MM implies that the Axiom of Determinacy holds for projective sets of reals.

Another argument in favor of forcing axioms is the following. There are a few nice examples of theorems (see for example [54]) discovered assuming forcing axioms and later obtained without these extra-assumptions, exactly as it occurred for Borel determinacy with respect to large cardinal axioms.

The possibility to discover new theorems is of course a good argument in favor of a new principle and, as noted before, assimilates justification in set theory to that of empirical sciences.

We can sum up the current situation of this first program as follows. Forcing axioms, such as MM, are non-constructive principles which are natural strenghtenings of Baire's category theorem and of AC; these axioms give strong effective means to settle problems formalizable in third order arithmetic. These means are also complemented by generic absoluteness results for third order arithmetic. The situation mirrors to a large extent that of second order arithmetic, in which AD and large cardinal axioms give strong means to answer many questions formalizable in second order arithmetic and are complemented by Woodin's generic absoluteness results.

However almost nothing is known on what the forcing axioms for fourth order arithmetic could be, or—more or less equivalently—what the largest class of compact Hausdorff spaces X for which $\mathsf{FA}_{\aleph_2}(X)$ holds could be. An exhaustive non-technical account on these matters can be found in [62].

Finally let us mention that there are also philosophical arguments that argue for a conceptual similarity between the notion of genericity connected to the method of forcing and the notion of *quasi-combinatorialism* introduced by Bernays [4], in discussing the foundations of set theory. In this context it is argued that forcing axioms are principles giving a precise mathematical instantiation of the, necessarily vague, notion of arbitrary set connected to quasi-combinatorialism [18, 57, 17].

6.2. **Ultimate-L.** An alternative program, orthogonal to forcing axioms, is proposed by Woodin and goes under the name of Ultimate-L. The strategy of the Ultimate-L program is not meant to offer a step-by-step completion of the theories of the initial segments of the universe of all sets, but instead aims at finding properties that could offer a global, detailed, picture of V.

The starting point is the observation that assuming $V = L$ one gets a nice "complete" picture of the universe of sets. For example all problems mentioned in this paper which are undecidable on the basis of ZFC gets an answer assuming $V = L$, that is CH holds, all Whitehead groups are free, there are outer automorphisms of the Calkin algebra, etc. However the axiom $V = L$ has serious drawbacks, in particular it is not compatible with all large cardinal axioms. This is the content of a theorem proved by Scott in 1961 [48] on the non existence of measurable—or Woodin, or supercompact, and all stronger—cardinals in L. It is also possible to show the incompatibility of L with the generic absoluteness results which follow from large cardinals. Indeed in L there are projective sets which do not have the Baire property, are not Lebesgue measurable, and do not have the perfect set property. Moreover it is possible to produce forcing extensions of L which give a different solution, with respect to the one computed in L, to problems formalizable in second order arithmetic.

Roughly speaking Woodin's program aims to devise a more comprehensive version of Gödel's constructible universe L; the so-called $\mathsf{Ultimate\text{-}L}$. The aim is to produce a model of set theory which retains the nice features of L, and avoids its unpleasant drawbacks. The definition of $\mathsf{Ultimate\text{-}L}$ and its analysis requires a technical background in set theory far above the threshold we put on this expository paper, hence our description of this program will be elusive, and will just try to give some basic ideas.

In the model $\mathsf{Ultimate\text{-}L}$ one retains the fine analysis of the universe of all sets offered by L. In particular its definition can be given by a highly constructive procedure, much in the same way as it occurs for L. A key role in the definition/construction of $\mathsf{Ultimate\text{-}L}$ is played once again by universally Baire sets. These are instrumental in the definition of the structures $HOD^{L(A,\mathbb{R})}$, where the parameter A stands for a universally Baire set. $L(A,\mathbb{R})$ is the model obtained by closing off with respect to Gödel operations the class $\mathsf{Ord} \cup \mathbb{R} \cup \{A\}$, and $HOD^{L(A,\mathbb{R})}$ is the ZFC-model given by hereditarily ordinal definable sets[48] in $L(A,\mathbb{R})$. Woodin's program is centered around the following axiom, called $\mathsf{V} = \mathsf{Ultimate\text{-}L}$.

> There are a proper class of Woodin cardinals[49]. Moreover, whenever φ is a sentence holding in an initial segment of the cumulative hierarchy of V, there is a universally Baire set A such that φ holds in some initial segment of $HOD^{L(A,\mathbb{R})}$.

Under $\mathsf{V} = \mathsf{Ultimate\text{-}L}$, Woodin offered a complete and detailed picture of the structure of the universe of all sets, much alike the picture one gets by analyzing the constructible universe L. In particular CH holds in any structure satisfying this principle. The axiom $\mathsf{V} = \mathsf{Ultimate\text{-}L}$ computes also the solution of many, if not all, of the undecidable problems mentioned in this paper, in most cases providing an answer opposite to that given by forcing axioms. Woodin's axiom also gives a global description of V, entailing that $V = HOD$, i.e. V is equal to the the class of hereditarily ordinal definable sets of V. Moreover, it can also be argued that $\mathsf{Ultimate\text{-}L}$ satisfies a minimality property similar to that of Gödel's L, since another non-trivial property of Woodin's axiom is that V cannot be obtained by forcing over some smaller submodel.

The name $\mathsf{Ultimate\text{-}L}$ is evocative, but its appropriateness still depends on the possibility to show its compatibility with large cardinal axioms. This is the content of the so called $\mathsf{V} = \mathsf{Ultimate\text{-}L}$ conjecture. A positive solution of this conjecture would be, in Woodin's opinion, the culmination of an important line of research, called *Inner model program*, which tries to build

[48] $X \in L(A,\mathbb{R})$ is ordinal definable in $L(A,\mathbb{R})$ if there is a formula $\phi(x,y)$ and an ordinal α such that $L(A,\mathbb{R}) \models \phi(X,\alpha)$ and $L(A,\mathbb{R}) \models \exists! x\phi(x,\alpha)$. $X \in L(A,\mathbb{R})$ is hereditarily ordinal definable in $L(A,\mathbb{R})$ if X and all the elements in its transitive closure are ordinal definable in $L(A,\mathbb{R})$.

[49] This is a technical requirement in order to obtain all regularity properties for the universally Baire sets.

canonical models of set theory that display similarities with L, but that nonetheless are compatible with all known large cardinals.

The $\mathsf{V} = \mathsf{Ultimate\text{-}L}$ conjecture is still open, although there are speculative arguments suggesting that it might not be possible to prove it. Indeed, the following interesting connection between large cardinals and Cohen's forcing method has been observed. Namely, if from large cardinal axioms it can be proved that a sentence is forcibly necessary, then it is often the case that higher large cardinals can prove that the same sentence holds in V. For a sentence φ, being *forcibly necessary* means that we can use forcing to build a model of ZFC where φ holds, and such that we cannot apply forcing once again, over that model, in order to falsify φ. An example of this phenomenon is given by the sentence asserting that all projective sets are Lebesgue measurable. Indeed, assuming the existence of a strong cardinal, this sentence is forcibly necessary, while assuming the existence of infinitely many Woodin cardinals it is just true; notice that if δ is a Woodin cardinal $(V_\delta, \in, =)$ models that there is a strong cardinal. Now, it has been shown that under large cardinals the negation of $\mathsf{V} = \mathsf{Ultimate\text{-}L}$ is forcibly necessary. Therefore, if the phenomenon we described above generalizes, it could give means to disprove the $\mathsf{V} = \mathsf{Ultimate\text{-}L}$ conjecture. Of course these are just mere speculations, only actual proofs can give definite answers.

In our opinion if the $\mathsf{V} = \mathsf{Ultimate\text{-}L}$ conjecture is true, there are good "extrinsic" reasons to accept Woodin's axiom $\mathsf{V} = \mathsf{Ultimate\text{-}L}$, given the nice picture of the universe of sets it provides, its ability to settle almost all questions which remain undecidable on the basis of ZFC, and its compatibility with large cardinal axioms.

It is unclear at this stage if these two programs can succeed and, if so, whether they will be able to refute each other beyond any reasonable doubt, or coexist harmoniously as it occurred finally for the Axiom of Choice and Determinacy hypotheses. What is striking is the effort of both programs in keeping up with Hilbert's rationalistic belief in the possibility to find a clear solution to all mathematical problems. This belief, far from being an unjustified hope, represents in our opinion the very essence of the study of the infinite and contrary to Kronecker's dictum, this study has proved to be fruitful and rewarding.

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