THE MODEL COMPANIONS OF SET THEORY

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Abstract. We first show that the first order theory of H_{ω_1} is the model companion of the first order theory of the universe of sets assuming the existence of class many Woodin cardinals, and working in a signature with predicates for all universally Baire sets of reals. We conclude our analysis with some basic conditions granting the model completeness of the first order theory of H_{ω_2} and of the axiom system $\mathsf{ZF} + V = L$ in an appropriate language.

Introduction

This paper outlines a deep connection between two important threads of mathematical logic: the notion of model companionship, a central concept in model theory due to Robinson, and the notion of generic absoluteness, which plays a fundamental role in the current meta-mathematical investigations of set theory.

In order to unveil this connection, we proceed as follows: we enrich the first order language in which to formalize set theory by predicates whose meaning is as "clear" as that of the $\in$-relation, specifically we add predicates for Δ_0-formulae and predicates for universally Baire sets of reals[1]. In this extended language we are able to apply Robinson's notions of model completeness and model companionship to argue that (assuming large cardinals) the first order theory of H_{ω_1} (the family of all hereditarily countable sets) is model complete and is the model companion of the first theor theory of V (the universe of all sets).

The study of model companionship goes back to the work of Abraham Robinson from the period 1950–1957 [9], and gives an abstract model-theoretic characterization of algebraically closed fields. Robinson introduced the notion of model completeness to characterize the closure properties of algebraically closed fields, and the notion of model companionship to describe the relation existing between these fileds and the rings without zero-divisors. Robinson then showed how to extend these notions and results to a variety of other classes of first order structures.

On the other hand, generic absoluteness characterizes exactly those set theoretic properties whose truth value cannot be changed by means of forcing.

In [11] we found the first indication of a strict connection existing between these two apparently unrelated concepts. In this paper we will enlighten this connection much further.

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[1] It is a standard result of set theory that Δ_0-formulae define absolute properties for transitive models of ZFC. On the other hand the notion of universal Baireness captures exactly those sets of reals whose first order properties cannot be changed by means of forcing (for example all Borel sets of reals are universally Baire). Therefore these predicates have a meaning which is clear across the different models of set theory. We do not expand further on this matter here, we just remark that: on the one hand a fine classification of which sets of reals are universally Baire and which are not would bring us into rather delicate grounds; on the other hand the results of this paper are based on the closure under first order definability of the class of universally Baire sets (i.e. closure under projections, finite intersections, finite unions, complementation), which is the case if we assume the existence of class many Woodin cardinals.

Recall that a first order theory T in a signature τ is model-complete if whenever $\mathcal{M} \sqsubseteq \mathcal{N}$ are models of T with one a substructure of the other, we get that $\mathcal{M} \prec \mathcal{N}$; i.e. being a substructure amounts to be an elementary substructure.

The theory of algebraically closed fields has this property, as it occurs for all theories admitting quantifier-elimination, however it is the case that many natural theories not admitting quantifier-elimination are model-complete. Robinson regarded model-completeness as a strong indication of tameness for a first order theory.

A weak point of this notion is that model completeness of a theory is very sensitive to the signature in which the theory is formalized: for all theories T in a signature τ there is a conservative extension to a theory T' in a signature τ' which admits quantifier elimination (it suffice to add symbols and axioms for Skolem functions to τ and T, [10, Thm. 5.1.8]). In particular we can always extend a first order language τ to a language τ' so to make a τ-theory T model-complete with respect to τ'. However if model-completeness of T is shown with respect to a "natural" language in which T can be formalized, then it brings many useful informations on the combinatorial-algebraic properties of models of T.

Recall also that for a first order signature τ, a τ-theory T is the model companion of a τ-theory S if T is model complete, and every model of T can be embedded in a model of S and conversely.

Robinson's infinite forcing is loosely inspired by Cohen's forcing method and gives an elegant formulation of the notion of model companionship: a theory T is the model companion of a theory S in the same first order signature if it is model complete and the models of T are exactly the infinitely generic structures for Robinson's infinite forcing applied to models of S. In [11] we find a fundamental connection between the notion of being an infinitely generic structure and that of being a structure satisfying certain types of forcing axioms. This suggests an interesting parallel between a semantic approach *à la Robinson* to the study of the models of set theory and generic absoluteness results.

The main result of this paper (Thm. 5.4) shows that, modulo a natural extension of the language of set theory (given by the addition of predicates for all universally Baire sets of reals), the existence of class many Woodin cardinals implies that the model companion of the theory of the universe of all sets is the theory of H_{ω_1}. We consider our extension natural because the predicates so added are exactly those whose truth value is unaffected by the forcing method, and for which, therefore, we have a concrete and stable understanding of their behaviour; for example Borel sets of reals are universally Baire, all sets of reals defined by a Δ_0-formula are universally Baire, and (assuming large cardinals) all universally Baire sets of reals have all the desirable regularity properties such as: Baire property, Lebesgue measurability, perfect set property, determinacy, etc.

We also remark that:

- On the one hand Hirschfeld [4] showed that ZF have a model companion in the signature $\{\in\}$. His result however is uninformative (a consideration he himself made in [4]), since the model companion of ZF for the signature $\{\in\}$ turns out to be (a small variation of) the theory of dense linear orders, a theory for a binary relation which has not much to do with the true meaning of the $\in$-relation. We consider this fact another indication of the naturalness of our choice of the first order language in which we choose to formalize set theory: in a first order language containing just the $\in$-relation, there are many basic concepts whose formalization in first order logic is syntactically too complex (for example being a surjective function is a Δ_0-property, but it is only Π_2-expressible in the signature $\{\in\}$), this discrepancy causes the "anomaly" of Hirschfeld's result, which is here resolved by adding predicates for all the concepts which are sufficiently simple and stable across the different models of set theory, i.e. the Δ_0-properties and the universally Baire predicates.

- On the other hand (unlike Hirschfeld's result) our results have a highly non-constructive flavour and require to embrace a fully platonistic perspective on the onthology of sets to be meaningfully formulated: we assume that the universe of sets V and the family of hereditarily countable sets H_{ω_1} are rightful elements of our semantics, that, whenever endowed with suitably defined predicates and constants, are well-defined first order structures for the appropriate signature. Of course it is possible to reformulate our results so to make them compatible with a formalist approach to set theory *à la Hilbert*, but in this case their meaning would be much less transparent, hence we refrain here from pursuing this matter further.

The main philosophical thesis we draw form the results of the present paper is that the success of large cardinals in solving problems of second-order arithmetic[2] via determinacy is due to the fact that these axioms make (in the appropriate language) the theory of H_{ω_1} the model-companion of the theory of V, and in particular a model complete theory.

Similar considerations can be drawn for other axioms (such as forcing axioms or the constructibility axiom $V = L$) which are able to decide most of the problems which cannot be settled on the basis of ZFC alone. In particular we show that if one has a simply definable well-order of H_{ω_2} (which is the case assuming the bounded proper forcing axioms hold), then one has simply definable Skolem functions producing witnesses of Δ_0-properties. In which case one can easily prove that the first order theory of H_{ω_2} is the model companion of the universe of sets in a signature with parameters for all elements of H_{ω_2}, predicates for all bounded formulae, and Skolem functions for such predicates. We prove as well that $\mathsf{ZFC} + V = L$ is model complete with respect to a natural appropriate first order language.

The paper is structured as follows:

- §1 recalls few important results on boolean-valued structures and generic absoluteness.
- §2 recalls the basic facts on model companionship and on Robinson's infinite forcing.
- In §3 we perform and justify the extension of the first order language of set theory, roughly described above, so to include predicates for all Δ_0-formulae; after relativizing the notion of model completeness to the generic multiverse, Theorem 3.6 shows that (assuming large cardinals) the theory of H_{ω_1} is the model companion of the theory of V relative to the generic multiverse for the language admitting predicates for all Δ_0-formulae.
- In §4 we offer reasons for the necessity of a further expansion of the language of set theory, which includes all universally Baire predicates.
- §5 gives the proof of Theorem 5.4 showing that in a language admitting predicates for all the universally Baire sets, the theory of H_{ω_1} is the model companion of the theory of V, if we assume the existence of class many Woodin cardinals.
- §6 extends the above result to the theory of H_{ω_2} assuming forcing axioms, and to the theory $\mathsf{ZFC} + V = L$.

1. Boolean valued models and generic absoluteness

Our first aim is to outline which first order properties are first order invariant with respect to the forcing method. Toward this aim we recall some standard facts on boolean-valued models for set theory, giving appropriate references for the relevant proofs (in particular [2], or [14], the forthcoming [1], the notes [13]), we assume below that the reader is familiar with the basic theory of boolean valued models, else we invite him to consult one of the above references (for example [13, Chapter 4]).

[2]All problems of second order arithmetic are first order properties of H_{ω_1}.

Recall that V denotes the universe of all sets and for any complete boolean algebra $\mathsf{B} \in V$

$$V^{\mathsf{B}} = \left\{ \tau : \tau : X \to \mathsf{B} \text{ is a function with } X \subseteq V^{\mathsf{B}} \text{ a set} \right\}$$

is the boolean valued model for set theory generated by forcing with B.

V^{B} is endowed with the structure of a B-valued model for the language of set theory $\mathcal{L} = \{\in, \subseteq\}$, letting (see [13, Def. 5.1.1] for details)

$$\llbracket \tau_1 \in \tau_2 \rrbracket_{\mathsf{B}} = \bigvee_{\sigma \in \operatorname{dom}(\tau_2)} (\llbracket \tau_1 = \sigma \rrbracket_{\mathsf{B}} \wedge \tau_2(\sigma)), \tag{1}$$

$$\llbracket \tau_1 \subseteq \tau_2 \rrbracket_{\mathsf{B}} = \bigwedge_{\sigma \in \operatorname{dom}(\tau_1)} (\neg \tau_1(\sigma) \vee \llbracket \sigma \in \tau_2 \rrbracket_{\mathsf{B}}), \tag{2}$$

$$\llbracket \tau_1 = \tau_2 \rrbracket_{\mathsf{B}} = \llbracket \tau_1 \subseteq \tau_2 \rrbracket_{\mathsf{B}} \wedge \llbracket \tau_2 \subseteq \tau_1 \rrbracket_{\mathsf{B}}. \tag{3}$$

The boolean value $\llbracket \phi(\tau_1, \ldots, \tau_n \rrbracket_{\mathsf{B}}$ of formulae $\phi(x_1, \ldots, x_n)$ with assignment $\tau_1, \ldots, \tau_n$ are given according to the standard rules of boolean valued semantics (see for example [13, Section 4.1]), concretely atomic formulae of types $\tau_1 \; R \; \tau_2$ are given the boolean value $\llbracket \tau_1 \; R \; \tau_2 \rrbracket_{\mathsf{B}}$, the boolean operations allows to define the boolean value associated to a conjunction/disjunction/negation of formulae, completeness of B allows to define

$$\llbracket \exists x \phi(x, \vec{\tau}) \rrbracket_{\mathsf{B}} = \bigvee_{\sigma \in V^{\mathsf{B}}} \llbracket \phi(\sigma, \vec{\tau}) \rrbracket_{\mathsf{B}}.$$

The class of models we will analyze is given by the generic extensions of initial segments of V. To make this precise we need a couple of definitions.

Definition 1.1. Let B be a complete boolean algebra. and $\dot{\kappa} \in V^{\mathsf{B}}$ be such that $\llbracket \dot{\kappa}$ is a regular cardinal$\rrbracket_{\mathsf{B}} = 1_{\mathsf{B}}$. Given $\kappa \geqslant \mathsf{B}$ least regular cardinal in V such that $\llbracket \dot{\kappa} \leqslant \check{\kappa} \rrbracket = 1_{\mathsf{B}}$ and B is $< \kappa$-CC, let

$$H^{\mathsf{B}}_{\dot{\kappa}} = \left\{ \tau \in V^{\mathsf{B}} \cap H^V_\kappa : \llbracket \tau \text{ has transitive closure of size less than } \dot{\kappa} \rrbracket_{\mathsf{B}} = 1_{\mathsf{B}} \right\}$$

It can be shown that $\llbracket \tau_1 \in \tau_2 \rrbracket_{\mathsf{B}}$, $\llbracket \tau_1 = \tau_2 \rrbracket_{\mathsf{B}}$, $\llbracket \tau_1 \subseteq \tau_2 \rrbracket_{\mathsf{B}}$ are well defined B-valued relations on $H^{\mathsf{B}}_{\dot{\kappa}}$ making it a B-valued model, the interpretation of all formulae follow the same rules given for V^{B}, except that in evaluating quantifiers now we let σ range just over the appropriate domain $H^{\mathsf{B}}_{\dot{\kappa}}$.

It is the case that for all G V-generic for B

$$H^{\mathsf{B}}_{\dot{\kappa}}[G] = \left\{ \tau_G : \tau \in H^{\mathsf{B}}_{\dot{\kappa}} \right\} = H^{V[G]}_{\dot{\kappa}_G},$$

i.e. $H^{\mathsf{B}}_{\dot{\kappa}}$ is a canonical family of B-names to denote the $H^{V[G]}_{\dot{\kappa}_G}$ of the generic extension. A key property of V^{B} and and of the models $H^{\mathsf{B}}_{\dot{\kappa}}$ defined above is fullness.

Definition 1.2. A B-valued model $\mathcal{M}$ for the signature $\mathcal{L}$ is *full* if for any $\mathcal{L}$-formula $\phi(x_0, \ldots, x_n)$ and $\tau_1, \ldots, \tau_n \in \mathcal{M}$

$$\llbracket \exists x \phi(x, \tau_1, \ldots, \tau_n) \rrbracket^{\mathcal{M}}_{\mathsf{B}} = \llbracket \phi(\sigma, \tau_1, \ldots, \tau_n) \rrbracket^{\mathcal{M}}_{\mathsf{B}}$$

for some $\sigma \in \mathcal{M}$.

Fact 1.3. *V^{B} and $H^{\mathsf{B}}_{\dot{\kappa}}$ are full B-valued model for any cba B and any $\dot{\kappa} \in V^{\mathsf{B}}$ such that $\llbracket \dot{\kappa}$ is a regular cardinal$\rrbracket_{\mathsf{B}} = 1_{\mathsf{B}}$.*

Proof. See [13, Thm. 5.1.34] for the case of V^{B}. The same proof can be easily adapted for $H^{\mathsf{B}}_{\dot{\kappa}}$ since all the predense subsets needed in the proof have size less than the κ chosen for the definition of $H^{\mathsf{B}}_{\dot{\kappa}}$. ■

For any ultrafilter G on B and $\mathcal{M}$ any structure among V^{B} or $H_{\check{\kappa}}^{\mathsf{B}}$, $\mathcal{M}/_G$ stands for the class (or set) $\{[\tau]_G : \tau \in \mathcal{M}\}$, where $[\tau]_G = \left\{\sigma \in V^{\mathsf{B}} : [\![\sigma = \tau]\!]_{\mathsf{B}} \in G\right\}$. We make $\mathcal{M}/_G$ a first order structure for the language $\{\in, \subseteq\}$, letting $[\tau]_G \; R\, /_G [\sigma]_G$ if and only if $[\![\tau \; R \; \sigma]\!] \in G$ for R among $\in, \subseteq$.

The forcing theorem states that:

- [13, Thm 4.3.2, Thm 5.1.34] (Łoś theorem for full boolean valued models) For all ultrafilter G on B, $\tau_1, \ldots, \tau_n \in V^{\mathsf{B}}$, and $\phi(x_1, \ldots, x_n)$

$$(V^{\mathsf{B}}/G, \in /_G) \models \phi([\tau_1]_G, \ldots, [\tau_n]_G) \text{ if and only if } [\![\phi(\tau_1, \ldots, \tau_n)]\!]_{\mathsf{B}} \in G.$$

- The same conclusion holds with $H_{\check{\kappa}}^{\mathsf{B}}$ in the place of V^{B}.
- [13, Thm. 5.2.3] Whenever G is V-generic for B the map

$$[\tau]_G \mapsto \tau_G = \{\sigma_G : \exists b \in G \, \langle \sigma, b \rangle \in \tau\}$$

  is the Mostowski collapse of the class V^{B}/G defined in $V[G]$ onto $V[G]$ and its restriction to $H_{\check{\kappa}}^{\mathsf{B}}/_G$ maps the latter onto $H_{\check{\kappa}_G}^{V[G]}$.

When $\mathsf{B} \in V$ is a $< \kappa$-cc complete boolean algebra, then $[\![\check{\kappa} \text{ is a regular cardinal}]\!] = 1_{\mathsf{B}}$. Therefore $H_{\check{\kappa}}^{\mathsf{B}}$ is a canonical set of B-names which describes the H_κ of a generic extension of V by B. The choice to work with $H_{\check{\kappa}}^{\mathsf{B}}$, instead of V^{B}, is motivated also by the fact that the former is a set definable in V using the parameters B and κ, while the latter is just a definable class in parameter B.

Having defined the structures we will be interested in (the structures $H_{\check{\kappa}}^{\mathsf{B}}/_G$) we now turn to the definition of the relevant morphisms between them.

Definition 1.4. Given $i : \mathsf{B} \to \mathsf{C}$ complete homomorphism of complete boolean algebras, i extends to a map $\hat{i} : V^{\mathsf{B}} \to V^{\mathsf{C}}$ defined by transfinite recursion by

$$\hat{i}(\tau) = \left\{\langle \hat{i}(\sigma), i(b) \rangle : \langle \sigma, b \rangle \in \tau\right\}.$$

Given $\tau_1, \ldots, \tau_n \in V^{\mathsf{B}}$, $\phi(\tau_1, \ldots, \tau_n)$ is generically absolute for i if

$$i([\![\phi(\tau_1, \ldots, \tau_n]\!]_{\mathsf{B}}) = \left[\!\!\left[\phi(\hat{i}(\tau_1), \ldots, \hat{i}(\tau_n)\right]\!\!\right]_{\mathsf{C}}.$$

It is well known that Δ_1-properties[3] are generically absolute (see for example [1, Prop. 4.1.2]); but it can be argued that Σ_1-properties in real parameters are also generically absolute. Indeed, we can prove the following lemma.

Lemma 1.5. *Assume that $\phi(x, y)$ is a Δ_1-property. Let $i : \mathsf{B} \to \mathsf{C}$ be a complete homomorphism. Then $\exists x \phi(x, y) \wedge y \subseteq \hat{\omega}$ is generically absolute for i.*

Proof. [12, Lemma 1.2] states that $H_{\omega_1}^M \prec_{\Sigma_1} N$ for any M (eventually non-transitive) model of ZFC and any N superstructure of M obtained by forcing over M (i.e for some B in M such that M models B is a complete boolean algebra, and some $G \in St(\mathsf{B})$, we have that $N = (V^{\mathsf{B}})^M/_G$). Apply the Lemma to the case $M = V^{\mathsf{B}}/_{i^{-1}[G]}$ and $N = V^{\mathsf{C}}/_G$ for any $G \in St(\mathsf{C})$. Then conclude by the forcing theorem. ■

The following is a major achievement of Woodin [8, Thm 3.1.7], conveniently reformulated in a weaker form and in a slightly different terminology for the purposes of this paper.

Theorem 1.6. *In the presence of class many Woodin cardinals, the structures of the form $H_{\omega_1^{\mathsf{B}}}^{\mathsf{B}}/_G$ are all models of the theory $Th(H_{\omega_1}^V)$ with parameters for elements of $H_{\omega_1}^V$.*

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[3] I.e. properties which are extension at the same time of a Π_1-formula and of a Σ_1-formula

2. Model theoretic completions

In what follows we are interested in studying certain classes of first order structures in a given first order signature τ; we will be interested just in theories consisting of sentences. To fix notation, if T is a first order theory in the signature τ, $\mathcal{M}_T$ denotes the τ-structures which are models of T.

Definition 2.1. A theory T is *model complete* if for all models $\mathcal{M}$ and $\mathcal{N}$ of T we have that $\mathcal{M} \sqsubseteq \mathcal{N}$ ($\mathcal{M}$ is a substructure of $\mathcal{N}$) implies $\mathcal{M} \prec \mathcal{N}$ ($\mathcal{M}$ is an elementary substructure of $\mathcal{N}$).

Definition 2.2. Let τ be a first order signature and T be a theory for τ. Given two models $\mathcal{M}$ and $\mathcal{N}$ of a theory T

- $\mathcal{M}$ is *existentially closed* in $\mathcal{N}$ ($\mathcal{M} \prec_1 \mathcal{N}$) if the existential and universal formula with parameters in $\mathcal{M}$ have the same truth value in $\mathcal{M}$ and $\mathcal{N}$.
- $\mathcal{M}$ is existentially closed for T if it is existentially closed in all its τ-superstructures which are models of T.

$\mathcal{E}_T$ denotes the class of τ-models which are existentially closed for T.

Note that in general models in $\mathcal{E}_T$ need not be models[4] of T. Model completeness describes exactly when this is the case.

Lemma 2.3. *[10, Lemma 3.2.7] (Robinson's test) Let T be a theory. The following are equivalent:*

(1) T is model complete.
(2) $\mathcal{E}_T = \mathcal{M}_T$.
(3) Each τ-formula is equivalent, modulo T, to a universal τ-formula.

Model completeness comes in pair with another fundamental concept which generalizes to arbitrary first order theories the relation existing between algebraically closed fields and commutative rings without zero-divisors. As a matter of fact, the case described below occurs when T^* is the theory of algebraically closed fields in some fixed characteristic and T is the the theory of comutative rings with no zero divisors in the same characteristic.

Definition 2.4. Given two theories T and T^*, in the same language τ, T^* is the *model companion* of T if the following conditions holds:

(1) Each model of T can be extended to a model of T^*.
(2) Each model of T^* can be extended to a model of T.
(3) T^* is model complete.

The model companion of a theory does not necessarily exist, but, if it does, it is unique.

Theorem 2.5. *[10, Thm. 3.2.9] A theory T has, up to equivalence, at most one model companion T^*.*

Different theories can have the same model companion, for example the theory of fields and the theory of commutative rings with no zero-divisors which are not fields both have the theory of algebraically closed fields as their model companion.

Remark 2.6. Using the fact that a theory T is mutually consistent with its model companion T^*, i.e. the models of one theory can be extended to a model of the other theory and vice-versa, together with the fact that universal theories are closed under sub-models it is easy to show that a theory and its model companion agree on their universal sentences.

[4]For example let T be the theory of commutative rings with no zero divisors which are not algebraically closed fields. Then $\mathcal{E}_T$ is exactly the class of algebraically closed fields and no model in $\mathcal{E}_T$ is a model of T.

Notation 2.7. In what follows, given a theory T, $T_\forall$ denotes the collection of all Π_1-sentences which are logical consequences of T. Similarly $T_\exists$ and $T_{\forall\exists}$ denote, respectively, the Σ_1 and the Π_2-theorems of T.

Theorem 2.8. *Let T be a first order theory. If its model companion T^* exists, then*

(1) $T_\forall = T^*_\forall$.
(2) T^* is the theory of the existentially closed models of $T_\forall$.
(3) T^* is axiomatized by $T_{\forall\exists}$.

Possibly inspired by Cohen's forcing method, Robinson introduced what is now called Robinson's infinite forcing [5]. In this paper we are interested in a slight generalization of Robinson's definition which makes the class of models over which we define infinite forcing an additional parameter.

Definition 2.9. Given a class of structure $\mathcal{C}$ for a signature τ, *infinite forcing for $\mathcal{C}$* is recursively defined as follows for a τ-formula $\phi(x_1, \dots, x_n)$, a structure $\mathcal{M} \in \mathcal{C}$ with domain M and $a_1, \dots, a_n \in M$:

- For $\phi(x_1, \dots, x_n)$ atomic, $\mathcal{M} \Vdash_\mathcal{C} \varphi(a_1, \dots, a_n)$ if and only if $\mathcal{M} \models \varphi(a_1, \dots, a_n)$;
- $\mathcal{M} \Vdash_\mathcal{C} \varphi(a_1, \dots, a_n) \wedge \psi(a_1, \dots, a_n)$ if and only if $\mathcal{M} \Vdash_\mathcal{C} \varphi(a_1, \dots, a_n)$ and $\mathcal{M} \Vdash_\mathcal{C} \psi(a_1, \dots, a_n)$;
- $\mathcal{M} \Vdash_\mathcal{C} \varphi(a_1, \dots, a_n) \vee \psi(a_1, \dots, a_n)$ if and only if $\mathcal{M} \Vdash_\mathcal{C} \varphi(a_1, \dots, a_n)$ or $\mathcal{M} \Vdash_\mathcal{C} \psi(a_1, \dots, a_n)$;
- $\mathcal{M} \Vdash_\mathcal{C} \forall x \varphi(x, a_1, \dots, a_n)$ if and only if (expanding τ with constant symbols for all elements of M) $\mathcal{M} \Vdash_\mathcal{C} \varphi(a, a_1, \dots, a_n)$, for every $a \in M$;
- $\mathcal{M} \Vdash_\mathcal{C} \neg\varphi(a_1, \dots, a_n)$ if and only if $\mathcal{N} \not\Vdash_\mathcal{C} \varphi(a_1, \dots, a_n)$ for all $\mathcal{N} \in \mathcal{C}$ superstructures of $\mathcal{M}$.

Robinson's infinite forcing consider only the case in which $\mathcal{C} = \mathcal{M}_T$. We are interested in considering Robinson's infinite forcing also in case $\mathcal{C}$ is not of this type.

As in the case of Cohen's forcing, this method produces objects that are generic. In this case generic models.

Notation 2.10. Given a class of structure $\mathcal{C}$ for a signature τ A structure $\mathcal{M} \in \mathcal{C}$ is *infinitely generic for $\mathcal{C}$* whenever satisfaction and infinite forceability coincide: i.e., for every formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$, we have

$$\mathcal{M} \models \varphi(a_1, \dots, a_n) \iff \mathcal{M} \Vdash_\mathcal{C} \varphi(a_1, \dots, a_n).$$

By $\mathcal{F}_\mathcal{C}$, we indicate the class of infinitely generic structures for $\Vdash_\mathcal{C}$.

Generic structures capture semantically the syntactic notion of model companionship.

Theorem 2.11. *Let T be a theory in a signature τ. The following are equivalent:*

(1) T^* exists.
(2) $\mathcal{E}_T$ is an elementary class.
(3) $\mathcal{F}_T$ is an elementary class.
(4) $\mathcal{E}_T = \mathcal{F}_{\mathcal{M}_{T_\forall}}$ (i.e. the existentially closed structures for T are the generic structures for Robinson's infinite forcing applied to the class $\mathcal{M}_{T_\forall}$).

3. The model companion of set theory for the generic multiverse

We already outlined that the model completeness of a theory is sensitive to the language in which that theory is expressed. We now embark in the task of selecting the right first order language to use for the construction of the model companion of (extensions of) ZFC. We will first argue that (at least for our purposes) this is neither the language $\{\in\}$ nor the language $\{\in, \subseteq\}$, even if these are the languages in which set theory is usually formalized in almost all textbooks.

As a preliminary result, we have that the model companion of ZF for the language $\{\in\}$ has been already fully described.

Theorem 3.1. *(Hirschfeld [4, Thm. 1, Thm. 5]) The universal theory of any $T \supseteq \mathsf{ZF}$ in the signature $\{\in\}$ is the theory*

$$S = \{\forall x_1 \dots \forall x_n (x_1 \notin x_2 \vee x_2 \notin x_3 \vee \dots \vee x_{n-1} \notin x_n \vee x_n \notin x_1) : n \in \mathbb{N}\}.$$

Letting for $A \subseteq n$

$$\delta_A(x_1, \dots, x_n, y) = \bigwedge_{i \in A} x_i \in y \wedge \bigwedge_{i \notin A} x_i \notin y,$$

the model companion of ZF is the theory

$$\begin{aligned} S^* = & \{\forall x_1 \dots x_n \exists y \, \delta_A(x_1, \dots, x_n, y) : n \in \omega, \, A \subseteq n\} \cup \\ & \cup \{\forall x, y \, \exists z [x = y \vee (x \in z \wedge z \in y) \vee (y \in z \wedge z \in x)]\} \end{aligned}$$

Notice that S only says that the graph of the $\in$-relation has no loops, while Hirschefeld also shows that in every model of S^* the interpretation of $\in$ defines a dense linear order without endpoints [4, Thm. 3]. In particular there is no apparent relation between the meaning of the $\in$-relation in a model of ZF (in its standard models it is a well-founded relation not linearly ordered) and the meaning of the $\in$-relation in models of S^* (it is a dense linear order without end-points).

We believe, that the above result, by Hirschfeld, gives a clear mathematical insight of why the language $\{\in\}$ is not expressive enough to describe the "right" model companion of set theory.

A key issue is the following: we are inclined to consider concepts and properties which can be formalized by formulae with bounded quantifiers much simpler and concrete than those which can only be formalized by formulae which make use of unrestricted quantification. This is reflected by the fact that properties formalizable by means of formulae with bounded quantifiers are absolute between transitive models of ZFC. This fact fails badly for properties defined by means of unbounded quantification.

For example the property f *is a function* is expressible using only bounded quantification, while the property κ *is a cardinal* is not. It is well known that the former is a property that is absolute between transitive models of ZFC containing f, while the latter is not. It is also a matter of fact that absolute properties are regarded as "tame" set theoretic properties (as their truth value cannot be changed by forcing, e.g f *is a function* remains true in any transitive model to which f belongs), while non-absolute ones are more difficult to control (they are not immune to forcing, e.g whenever κ is an uncountable cardinal of the ground model, it will not be anymore so in a generic extension by $\mathrm{Coll}(\omega, \kappa)$).

Hence it is necessary to formalize set theory in a first order language able to recognize syntactically the different semantic complexity of absolute and non-absolute concepts. As Hirschfeld has shown this is not the case for the ZF-axioms in the language $\{\in\}$.

In Kunen and Jech's books the solution adopted is that of passing from first order logic to a logic with bounded quantifiers $\exists x \in y$ and $\forall x \in y$ binding the variable x so that $\exists x \in y \phi(x, y, \vec{z})$ is logically equivalent to $\exists x (x \in y \wedge \phi(x, y, \vec{z}))$ and $\forall x \in y \phi(x, y, \vec{z})$ is logically equivalent to $\forall x (x \in y \rightarrow \phi(x, y, \vec{z}))$. In this new logic f *is a function* is expressible by a formula with only bounded quantifiers, while κ *is a cardinal* is expressible by a formula of type $\forall x \phi(x, \kappa)$ with ϕ having only bounded quantifiers. On the other hand Jech and Kunen's solution is not convenient for the scopes of this paper, because it formalizes set theory outside first order logic, making less transparent how we could use model theoretic techniques (designed expressly for first order logic) to isolate what is the correct model companion of set theory. The alternative solution we adopt in this paper is that of expressing set theory in a first order language with relation symbols for any bounded formula.

Definition 3.2. Given the first order signature

$$\mathcal{L}^* = \{R_\phi : \phi \text{ logically equivalent to a bounded formula in the signature } \{\in\}\},$$

ZFC^* is the $\mathcal{L}^*$-theory obtained adding to ZFC the axioms

$$\forall \vec{x}(\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x}))$$

for all formulae $\phi(\vec{x})$ logically equivalent to a bounded formula.

In ZFC^* we now obtain that many absolute concepts (such as that of being a function) are now expressed by an atomic formula, while certain more complicated ones (for example those defined by means of transfinite recursion over an absolute property, such as *x is the transitive closure of y*) can still be expressed by means of Δ_1-properties of $\mathcal{L}^*$ (i.e. properties which are formalized at the same time by a Π_1-formula and by a Σ_1-formula), hence are still absolute between any two models (even non-transitive) $\mathcal{M}$, $\mathcal{N}$ of ZFC^* of which one is a substructure of the other. On the other hand many definable properties have truth values which may vary depending on which model of ZFC^* we work in (for example *κ is an uncountable cardinal* is a $\Pi_1 \setminus \Sigma_1$-property in ZFC^* whose truth value may depend on the choice of the model of ZFC^* to which κ belongs).

Our first aim is to identify what is $\mathsf{ZFC}^*_\forall$. To this aim, first recall that Levy's absoluteness gives that $H_{\omega_1} \prec_{\Sigma_1} V$, and that for any set X there is a forcing extension in which X is countable (just force with $\mathrm{Coll}(\omega, X)$). In particular one can argue that the Π_2-assertion $\forall X \exists f : \omega \to X$ *surjectve* is generically true for Robinson's infinite forcing applied to the forcing extensions of V. Notice that $H_{\omega_1} \models \forall X \exists f : \omega \to X$ *surjectve*.

The natural conjecture is to infer that the first order theory of H_{ω_1} is the model companion of the first order theory of V. We now show exactly to which extent the conjecture is true, while proving that it is false.

We first relativize the notion of model completeness to this new setting.

Definition 3.3. Given a theory T and a category $(\mathbb{M}, \to_{\mathbb{M}})$ with $\mathbb{M}$ a class of models of T and $\to_{\mathbb{M}}$ a class of morphisms between them, T is *model complete with respect to* $\mathbb{M}$ if for all models $\mathcal{M}$ and $\mathcal{N}$ in $\mathbb{M}$ we have that $\mathcal{M} \prec \mathcal{N}$, whenever there is a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\to_{\mathbb{M}}$.

In order to define the class of structures and morphisms $\mathbb{M}$ we need the following useful results (see for example [1, Prop. 4.1.2])

Proposition 3.4. *Let B and C be complete boolean algebras.*

(1) Given $k : \mathsf{B} \to \mathsf{C}$ complete homomorphism of complete boolean algebras, define

$$\hat{k} : V^{\mathsf{B}} \to V^{\mathsf{C}}$$

by transfinite recursion letting

$$\hat{k}(\sigma) = \left\{\langle \hat{k}(\tau), k(b)\rangle : \langle \tau, b\rangle \in \sigma\right\}.$$

Then for any Δ_1-property $P(x_1, \ldots, x_n)$ in $\mathcal{L}^*$ and every $\tau_1, \ldots, \tau_n \in V^{\mathsf{B}}$

$$k(\llbracket P(\tau_1, \ldots, \tau_n)\rrbracket_{\mathsf{B}}) = \left\llbracket P(\hat{k}(\tau_1), \ldots, \hat{k}(\tau_n))\right\rrbracket_{\mathsf{C}}.$$

(2) Moreover whenever $f : \mathsf{B} \to \mathsf{C}$ is a complete homomorphism, for any $H \in St(\mathsf{C})$ such that such that $\left\llbracket \hat{f}(\dot{\kappa}) \leqslant \dot{\delta}\right\rrbracket_{\mathsf{C}} \in H$, letting $G \in St(\mathsf{B})$ be $f^{-1}[H]$, the map

$$\begin{aligned} \hat{f}/_H : & H^{\mathsf{B}}_{\dot{\kappa}}/_G \to H^{\mathsf{C}}_{\dot{\delta}}/_H \\ & [\tau]_G \mapsto [\hat{f}(\tau)]_H \end{aligned}$$

is an $\mathcal{L}^*$-morphism.

Definition 3.5. The *generic multiverse* $(\Omega(V), \to_{\Omega(V)})$ is the collection:

$$\left\{H^{\mathsf{B}}_{\dot\kappa}/_G : [\![\dot\kappa \text{ is a regular cardinal}]\!]_{\mathsf{B}} = 1_{\mathsf{B}},\, G \in St(\mathsf{B})\right\}.$$

its morphism are the $\mathcal{L}^*$-morphisms of the form $\hat f/_H : H^{\mathsf{B}}_{\dot\kappa}/_G \to H^{\mathsf{C}}_{\dot\delta}/_H$ for some complete homomorphism $f : \mathsf{B} \to \mathsf{C}$ with $H \in St(\mathsf{C})$, $G = f^{-1}[H]$, $\left[\!\!\left[\hat f(\dot\kappa) \leqslant \dot\delta\right]\!\!\right]_{\mathsf{C}} \in H$.

Notice[5] that $\Omega(V)$ is a definable class in V. $\Omega(V)$ is a formulation in the language of boolean valued models of the notion of generic multiverse.

This is the first result we want to bring forward:

Theorem 3.6. *The first order theory with parameters for elements of* $H^V_{\omega_1}$ *of the* $\mathcal{L}^*$-*structure* $(H^V_{\omega_1}, R^V_\phi : R_\phi \in \mathcal{L}^*)$ *in the signature* $\mathcal{L}^* \cup H_{\omega_1}$ *is the model companion of* ZFC^*+ there exist class many Woodin cardinals *with respect to* $(\Omega(V), \to_{\omega(V)})$.

Proof. We prove the Theorem in a series of lemmas. By ω_1^{B} we denote a B-name such that

$$[\![\omega_1^{\mathsf{B}} \text{ is the first uncountable cardinal}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}.$$

Lemma 3.7. $H^{\mathsf{B}}_{\omega_1^{\mathsf{B}}}/_G$ *is existentially closed with respect to its superstructures in* $\Omega(V)$.

Proof. It is a reformulation of Cohen's absoluteness lemma, that is Lemma 1.5. ■

Fact 3.8. *Given any structure in* $\Omega(V)$, *there is a natural morphism that embeds it in a structure of the form* $H^{\mathsf{C}}_{\omega_1^{\mathsf{C}}}/_G$.

Proof. Let $H^{\mathsf{B}}_{\dot\kappa}/_G \in \Omega(V)$. Find a regular $\delta > 2^\kappa$ and consider the forcing notion $\mathrm{Coll}(\omega, < \delta)$. By a classical forcing result (see for example [6, Lemma 26.9]), we have that B is isomorphic to a complete sub-algebra of the boolean completion C of $\mathrm{Coll}(\omega, < \delta)$, i.e there is a (even injective) complete homomorphism $f : \mathsf{B} \to \mathsf{C}$.

Moreover it is well known (see [6, Thm. 15.17(iii)]) that

$$\left[\!\!\left[\check\delta \text{ is the first uncountable cardinal}\right]\!\!\right]_{\mathsf{C}} = 1_{\mathsf{C}}.$$

Hence $\omega_1^{\mathsf{C}} = \check\delta$.

Extend the prefilter $f[G]$ on C to an ultrafilter H on C. Then $f^{-1}[H] = G$. Since $H^{\mathsf{C}}_{\check\delta} = H_\delta \cap V^{\mathsf{C}}$ (by the $< \delta$-CC of C), and $\left[\!\!\left[\hat f(\dot\kappa) < \check\delta\right]\!\!\right]_{\mathsf{C}} = 1_{\mathsf{C}} \in H$, $\hat f[H^{\mathsf{B}}_{\dot\kappa}] \subseteq H^{\mathsf{C}}_{\check\delta}$. Hence $\hat f/_H$ is a morphism in $\to_{\Omega(V)}$. ■

This completes the proof of Theorem 3.6: by Theorem 1.6 the models of $\mathrm{Th}(H^V_{\omega_1})$ with parameters for elements of $H^V_{\omega_1}$ in $\Omega(V)$ are of the form $H^{\mathsf{B}}_{\omega_1^{\mathsf{B}}}/_G$. By Lemma 3.7 $\mathrm{Th}(H_{\omega_1})$ is model complete with respect to $\Omega(V)$. Finally Fact 3.8 provides the mutual consistency between arbitrary models in $\Omega(V)$ and models of $\mathrm{Th}(H_{\omega_1})$ in $\Omega(V)$.

■

Two natural questions arise:

- is the $\mathcal{L}^*$-theory $T = \mathrm{Th}(\langle H^V_{\omega_1}, R^V_\phi : R_\phi \in \mathcal{L}^*, H_{\omega_1}\rangle)$ model complete?
- Can we embed any set sized model $\mathcal{L}^*$-model of $S = \mathrm{Th}(\langle V, R^V_\phi : R_\phi \in \mathcal{L}^*, H_{\omega_1}\rangle)$ into some model of $\Omega(V)$ and conversely?

[5]There can be morphisms $h : H^{\mathsf{B}}_{\kappa}/_G \to H^{\mathsf{C}}_{\delta}/_H$ which are not of the form $\hat f/_H$ for some complete homomorphism $f : \mathsf{B} \to \mathsf{C}$, even in case B preserve the regularity of κ and C the regularity of δ. We do not spell out the details of such possibilities.

If we could answer positively to both questions we would have that T is the model companion of $\mathrm{Th}(\langle V, R_\phi^V : R_\phi \in \mathcal{L}^*, H_{\omega_1}\rangle)$, since H_{ω_1} is Σ_1-elementary in V with respect to $\mathcal{L}^*$, hence the two structures have the same universal theory and we can apply Robinson's test to the two theories.

In the forthcoming [15] the second author and Parente show that the answer to the second question is positive (assuming large cardinal axioms). This is already quite interesting: it outlines that any set sized $\mathcal{L}^*$-model of the theory of (an initial segment of) V (obtained by whatever means model theory gives us) is in fact a substructure of a $\mathcal{L}^*$-model of the theory of (an initial segment of) V obtained by forcing.

Nonetheless in the next section we argue that the first question has a negative answer. This will bring us to further expand the language of set theory, including predicates for universally Baire sets, in order to argue that Woodin's generic absoluteness results for this type of sets bring, as a byproduct, the model completeness of the theory of H_{ω_1} with predicates for the universally Baire sets.

4. Second order arithmetic and $\mathrm{Th}(H_{\omega_1})$

We define second order number theory as the first order theory of the structure

$$(\mathcal{P}(\mathbb{N}) \cup \mathbb{N}, \in, \subseteq, =, \mathbb{N}).$$

Π^1_n-sets (respectively Σ^1_n, Δ^1_n) are the subsets of $\mathcal{P}(\mathbb{N}) \equiv 2^{\mathbb{N}}$ defined by a Π_n-formula (respectively by a Σ_n-formula, at the same time by a Σ_n-formula and a Π_n-formula in the appropriate language), if the formula defining a set $A \subseteq (2^{\mathbb{N}})^n$ has some parameter $r \in 2^{\mathbb{N}}$ we accordingly say that A is $\Pi^1_n(r)$ (respectively $\Sigma^1_n(r)$, $\Delta^1_n(r)$).

Definition 4.1. Given $a \in H_{\omega_1}$, $r \in 2^{\mathbb{N}}$ codes a, if (modulo a recursive bijection of $\mathbb{N}$ with $\mathbb{N}^2$) we have that r codes a well-founded extensional relation on $\mathbb{N}$ whose transitive collapse is the transitive closure of $\{a\}$.

- Cod : $2^{\mathbb{N}} \to H_{\omega_1}$ is the map assigning a to r if and only if r codes a and assigning the emptyset to r otherwise.
- WFE is the set of $r \in 2^{\mathbb{N}}$ which (modulo a recursive bijection of $\mathbb{N}$ with $\mathbb{N}^2$) are a well founded extensional relation.

The following are well known facts[6].

Remark 4.2. The map Cod is defined by a provably Δ_1-property over H_{ω_1} and is surjective. Moreover WFE is a Π^1_1-subset of $2^{\mathbb{N}}$.

Lemma 4.3. *Assume $A \subseteq 2^{\mathbb{N}}$ is Σ^1_{n+1}. Then A is Σ_n-definable in H_{ω_1} in the language $\mathcal{L}^*$.* ■

Lemma 4.4. *Assume A is Σ_n-definable in H_{ω_1} in the language $\mathcal{L}^*$. Then $A = Cod^{-1}[Cod[A]]$, and $Cod[A]$ is Σ^1_{n+1}.*

We can now easily conclude the following:

Theorem 4.5. *$T = Th(\langle H_{\omega_1}, R_\phi^V : R_\phi \in \mathcal{L}^*, H_{\omega_1}\rangle)$ is not model complete.*

Proof. For all n there is some $A_n \in \Sigma^1_{n+1} \setminus \Pi^1_n$ (see for a proof [7, Thm. 22.4]). Therefore A_2 is Σ_2-definable but not Π_1-definable in H_{ω_1}. Consequently, Robinson test fails and T is not model complete. ■

---

[6] See [6, Section 25] and in particular the statement and proof of Lemma 25.25, which contains all ideas on which one can elaborate to draw the conclusions below.

5. Model completeness for set theory with predicates for universally Baire sets

Given a topological space (X, τ), $A \subseteq X$ is nowhere dense if its closure has a dense complement, meager if it is the countable union of nowhere dense sets, with the Baire property if it has meager symmetric difference with an open set.

Definition 5.1. (Feng, Magidor, Woodin) $A \subseteq 2^{\mathbb{N}}$ is *universally Baire* if for every compact Hausdorff space X and every continuous $f : X \to 2^{\mathbb{N}}$ we have that $f^{-1}[A]$ has the Baire property in X.

Theorem 5.2. *Let T be the $\mathcal{L}^*$-theory* ZFC^*+there are class many Woodin cardinals*.*

1. *[8, Thm. 3.3.9, Thm. 3.3.14] Assume V models T. Then every set of reals in $L(\mathbb{R})$ is universally Baire.*
2. *[8, Thm. 3.4.17] Assume $V \models T$ and is obtained as a generic extension of W such that for some δ which is supercompact in W, we have that $(2^{\delta})^W$ is countable in V. Let UB be the family of universally Baire sets in V. Then every subset of $2^{\mathbb{N}}$ in $L(\mathsf{Ord}^{\omega}, \mathsf{UB})^V$ is universally Baire.*

Theorem 5.3. *Let T be the theory* ZFC^*+there are class many Woodin cardinals*. Assume V models T and condition (2) of Thm. 5.2 holds. Let $\mathcal{L}^{**} = \mathcal{L}^* \cup \{B : \overline{B} \in \mathsf{UB}\}$. Then the $\mathcal{L}^{**}$-theory T_1 of*

$$\mathcal{M} = (H_{\omega_1}, \overline{R}_{\phi} : \ \phi \text{ bounded}, \overline{B} : \overline{B} \in \mathsf{UB})$$

is model complete.

Proof. Let $A \subseteq H_{\omega_1}$ be defined as the extension in $\mathcal{M}$ of some $\mathcal{L}^{**}$-formula $\phi(x, r_1, \ldots, r_n)$ with $r_i \in 2^{\mathbb{N}}$.

Then $B = \mathrm{Cod}^{-1}[A] \cap \mathsf{WFE}$ is a definable subset of $2^{\mathbb{N}}$ in

$$(H_{\omega_1}, \overline{R}_{\phi} : \ \phi \text{ bounded}, \overline{B} : \overline{B} \in \mathsf{UB}),$$

hence it belongs to $L(\mathsf{Ord}^{\omega}, \mathsf{UB})$, therefore $B \in \mathsf{UB}$.

Now

$$A = \{a \in H_{\omega_1} : \forall y(\langle y, a \rangle \in \mathrm{Cod} \to y \in B)\}.$$

Since $(\langle y, a \rangle \in \mathrm{Cod})$ can be expressed by a Σ_1-formula in the structure

$$(H_{\omega_1}, \overline{R}_{\phi} : \ \phi \text{ bounded}, \ \overline{B} : \overline{B} \in \mathsf{UB}),$$

we have that A is the extension of a Π_1-formula $\psi(x)$ using the universally Baire predicate B in the structure

$$(H_{\omega_1}, \overline{R}_{\phi} : \ \phi \text{ bounded}, \ \overline{B} : \overline{B} \in \mathsf{UB}).$$

By the third criterion of Robinson's test we conclude that T_1 is model complete. ■

Theorem 5.4. *Let T be the theory* ZFC+there are class many Woodin cardinals*. Assume V models T and condition (2) of Thm. 5.2 holds. Let:*

- *T_0 be the $\mathcal{L}^{**}$-theory of V with parameters in H_{ω_1} and predicates for all elements of* UB*,*
- *T_1 be the $\mathcal{L}^{**}$-theory with parameters of*

$$(H_{\omega_1}, \overline{R}_{\phi} : \ \phi \text{ bounded}, \overline{B} : \overline{B} \in \mathsf{UB}).$$

Then T_1 is the model companion of T_0.

Proof. By (a slight variation of the proof of) Levy's absoluteness we have that

$$(H_{\omega_1}, \overline{R}_{\phi} : \ \phi \text{ bounded}, \ \overline{B} : \overline{B} \in \mathsf{UB}) \prec_1 (V, \overline{R}_{\phi} : \ \phi \text{ bounded}, \ \overline{B} : \overline{B} \in \mathsf{UB}).$$

In particular T_1 and T_0 satisfy the same universal sentences.

It is now a standard result in model theory [10, Lemma 3.1.2] that in this case it is possible to embed any model $\mathcal{M}$ of each theory into some model $\mathcal{N}$ of the other theory by choosing $\mathcal{N}$ saturated enough so to realize all existential types of $\mathcal{M}$. The conclusion follows by model completeness of T_1.

■

Minimal variations of the above argument yield the following result:

Theorem 5.5. *Let T be the theory* ZFC+there are class many Woodin cardinals. *Assume V models T and condition (1) of Thm. 5.2 holds. Let:*

- *T_0 be the $\mathcal{L}^{**}$-theory of V with parameters in H_{ω_1} and predicates for all sets of reals definable in $L(\mathbb{R})$,*
- *T_1 be the $\mathcal{L}^{**}$-theory with parameters of*

$$(H_{\omega_1}, \overline{R}_\phi : \ \phi \text{ bounded}, \overline{B} : \overline{B} \in L(\mathbb{R}) \cap \mathcal{P}(2^\omega)).$$

Then T_1 is the model companion of T_0.

6. Model completeness for the theory of H_{ω_2} assuming forcing axiom and for the theory of V assuming $V = L$

We can show that mild forcing axioms such as the bounded proper forcing axiom BPFA already entail model completeness for the $\mathcal{L}^*$-theory of H_{ω_2} expanded by absolutely definable Skolem functions. Similarly we will argue that $\mathsf{ZFC}^* + V = L$ is model complete in the appropriate natural language. This is a rather straightforward consequence of the existence of simply definable well-orders of H_{ω_2} in the presence of forcing axioms and of a simply definable well order of L. We investigte in some details the model completeness of the theory of H_{ω_2} assuming forcing axioms and briefly discuss the model completeness of $\mathsf{ZFC} + V = L$ in the appropriate natural language.

We will use the following result:

Theorem 6.1 (Caicedo, Veličković)**.** *[3, Thm. 2] Assume* BPFA *and let $A \subseteq \omega_1$ be a ladder system on ω_1. There is a* ZF$\setminus$*Power-set-provably Δ_1-definable property $P(x, y, z)$ in the signature $\mathcal{L}^*$ such that $P(x, y, A)$ provides a well-order of H_{ω_2} in type ω_2.*

We now expand $\mathcal{L}^*$ to the signature $\mathcal{L}^{**}$ obtained adding constant symbols for ω, ω_1, H_{ω_2}, A, and a function symbol f_ϕ of ariety n_ϕ for each $R_\phi \in \mathcal{L}^*$ of ariety $n_\phi + 1$. We then extend ZFC^* to a $\mathcal{L}^{**}$ obtained by adding:

- The axiom (expressible in the signature $\mathcal{L}^* \cup \{H_{\omega_2}, \omega_1\}, \omega)$ stating that ω_1 is the first uncountable cardinal

$$\forall f (f \text{ is a function with domain } \omega \to \omega_1 \nsubseteq \operatorname{ran}(f)).$$

- The axiom (expressible in the signature $\mathcal{L}^* \cup \{H_{\omega_2}, \omega_1, \omega\}$) stating that H_{ω_2} is the set of all sets with transitive closure of size ω_1

$$\forall x (x \in H_{\omega_2} \leftrightarrow x \textit{ has transitive closure of size at most } \omega_1)$$

(remark that x *has transitive closure of size at most* ω_1 is a $\Delta_1(\omega, \omega_1)$-property for ZFC^*).
- The axiom (expressible in the signature $\mathcal{L}^* \cup \{A, \omega_1, \omega\}$)

$$A \subseteq \omega_1 \textit{ codes a ladder system on } \omega_1.$$

(A ladder system on ω_1 is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that $C_\alpha \subseteq \alpha$ and C_α is cofinal in α of order type ω whenever α is a limit ordinal).

- The axioms

$$\forall x_1 \dots x_{n_\phi}[$$
$$(\bigwedge_{i=1}^{n} x_i \in H_{\omega_2}) \to$$
$$[$$
$$(\forall y \neg R_\phi(y, x_1, \dots, x_{n_\phi}) \wedge f_\phi(x_1, \dots, x_{n_\phi}) = A)$$
$$\vee$$
$$(R_\phi(f_\phi(x_1, \dots, x_{n_\phi}), x_1, \dots, x_{n_\phi}) \wedge \forall u(P(u, f_\phi(x_1, \dots, x_{n_\phi}), A) \to \neg R_\phi(u, x_1, \dots, x_{n_\phi})))$$
$$]$$
$$]$$

stating that for any $x_1 \dots, x_n \in H_{\omega_2}$ $f_\phi(x_1, \dots, x_n)$ is the least element y such that $R_\phi(y, x_1, \dots, x_n)$ according to the well-order of H_{ω_2} defined by $P(x, z, A)$ (if such a y exists), and is A otherwise.

Remark that all the above axioms are universal statements in the language $\mathcal{L}^{**}$. We can immediately prove the following:

Theorem 6.2. *Let T be any complete extension of* ZFC** + BPFA *and M a model of T. Let S be the $\mathcal{L}^{**}$-theory of the structure $H_{\omega_2}^M$. Then S is the model companion of T.*

Proof. The axioms added to $\mathcal{L}^{**}$ yield that $H_{\omega_2}^M$ satisfies

$$\forall x_1 \dots x_{n_\phi}[$$
$$(\forall y \neg R_\phi(y, x_1, \dots, x_{n_\phi}) \wedge f_\phi(x_1, \dots, x_{n_\phi}) = A)$$
$$\vee$$
$$(R_\phi(f_\phi(x_1, \dots, x_{n_\phi}), x_1, \dots, x_{n_\phi}) \wedge \forall u(P(u, f_\phi(x_1, \dots, x_{n_\phi}), A) \to \neg R_\phi(u, x_1, \dots, x_{n_\phi})))$$
$$]$$

for all $R_\phi \in \mathcal{L}^*$. Therefore S admits quantifier elimination, and is a universal $\mathcal{L}^{**}$-theory, by well known standard results on the Skolemization of first order theories (see for example [10, Thm. 5.1.8, and proof of the Claim in Cor. 5.1.9]). We conclude that S is model complete (by quantifier elimination any substructure of a model N of S which is itself a model of S is an elementary substructure of N). Since $H_{\omega_2}^M$ and M satisfy the same universal $\mathcal{L}^*$-sentences[7]. We conclude by Robinson's test. ■

We have a series of remarks to make.

Remark 6.3. The above result does not say that ZFC** + BPFA has a model companion. For example assume $M \models$ BMM+*there exists a reflecting cardial* δ. Let N be the generic extension of M by standard proper forcing of length δ. Then M and N are both models of ZFC** + BPFA (since BMM implies BPFA). On the other hand in $H_{\omega_2}^M$ it holds that the family of canonical functions on ω_1 is dominating modulo club, while $H_{\omega_2}^N$ models that this family is not dominating. Hence $H_{\omega_2}^M$ is an $\mathcal{L}^{**}$-substructure of $H_{\omega_2}^N$ which is not elementary. The result just says that any *complete extension of* ZFC** + BPFA has a model companion.

In case one assumes $V = L$ we can produce a more constructive result:

[7]Notice that T does not admit quantifier elimination because the Skolemization fails for the n_ϕ-tuples $x_1, \dots, x_{n_\phi}$ which are not all in H_{ω_2}.

Theorem 6.4. *Consider the language $\mathcal{L}^* \cup \{<\}$, with $<$ a binary relation symbol. Let ψ be the $\mathcal{L}^* \cup \{<\}$-sentence asserting that $<$ defines one of the provably Δ_1-definable well-order of L. The theory $\mathsf{ZFC} + V = L + \psi$ is model complete with respect to $\mathcal{L}^* \cup \{<\}$.*

Proof. We leave the details to the reader. ■

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