

CANTOR AND THE INFINITE

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ABSTRACT. In this article we review Cantor's contribution to the construction of a theory of infinity. We will present and discuss the technical achievements and the philosophical ideas that brought Cantor to the creation of set theory and to its justification.

INTRODUCTION

The history of infinity is probably as long as the history of human thought. In its multifaceted aspects the concept of infinity has always amazed, frightened, or challenged the sharpest minds. The close connection between infinity and many different areas of human knowledge shows different, often frustrated, attempts to make this concept intelligible. In the effort of dominating the frightening abysses of infinity many contributions have been offered to philosophy, religion, mysticism, literature, and science.

Of course the smaller attempt to bring to unity these innumerable contributions to the history of human culture would fall short of the heterogeneous ideas and perspectives that animated the many brilliant minds that populated this story¹.

Although deeply connected to this general cultural development, the history of infinity in mathematics is easier to trace and to describe. Moreover, because a direct confrontation with infinity was avoided until very recently, much of this story can be safely considered pre-history. Indeed, the full acceptance of infinity in mathematics happened only in the second half of the XIX century and, to make the historical work even neater, it founded its main champion in a single man, Georg Cantor, who following the steps of Prometheus brought light to the mathematical analysis of infinity.

The development of ideas is way more complex than the necessary simplification operated by history, and there are mathematical and philosophical reasons to argue that this theory of infinity was not born

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¹However, we find in the literature some attempts [21], [6], [19].

from Cantor's mind as Athena from Zeus'. As accurately described in [9] a favorable idealistic cultural environment, together with the contributions and ideas of Riemann and Dedekind, paved the way for the acceptance of infinity, introducing into mathematical practice notions and concepts whose full-fledged justification needed the recognition of infinity as a proper mathematical concept.

However there is a clear difference between a practical acceptance and a theoretical defense. This is why we can safely consider Cantor as the main actor in the creation of a mathematical theory of infinity. His engagement and fundamental discoveries opened a new era for mathematics; one where infinity was not anymore considered as a *façon de parler*, but the central concept of mathematics; and one where its axiomatic counterpart, set theory, rapidly became the *lingua franca* of the mathematical community, thus realizing Leibniz's dream.

This Copernican revolution is what we will briefly outline in these pages. After a very short overview of the main concept and of the pre-Cantorian history of infinity, §1, we will present the contribution of Cantor to the theorization of infinity, in §2. Then in §3 we will present a more philosophical perspective on Cantor's view and on the justification of infinity, contextualizing Cantor's ideas with the foundations of mathematics of the end of the XIX century.

1. INFINITY IN MATHEMATICS

The concept of infinity is, explicitly or indirectly, at the center of much of the philosophical discussion in ancient Greece. Two forms of infinity were categorized and discussed—and this distinction would survive until Cantor. Following an Aristotelian terminology, infinity can be understood potentially or actually.

Potential infinity is assimilated to unboundedness. Something is potentially infinite if its size can always be augmented without this process never coming to an end. Natural numbers were seen as a potentially infinite collection of numbers, since the operator of adding one more will always give rise to larger and larger numbers.

Of course, by definition, the process of forming a potential infinite collection will never come to an end. However, if we abstract from the process and we consider only what results from it, that is the complete infinite totality of objects so produced, we arrive at the notion of *actual infinity*. The collection of all natural numbers build by the successor operation, considered as a whole complete collection, can thus be seen as an example of an actual infinite collection.

This distinction instantiates also another dichotomy, more philosophical, under which we can understand infinity: its twofold character both intensional and extensional. Indeed, viewed as a process, natural numbers could be described as the collection of numbers that are produced by the iteration of the successor operation, starting from zero. In other terms natural numbers could be captured *intensionally* giving the rules that allow the construction of this collection. But they can also be captured *extensionally*, specifying the objects that compose \mathbb{N} ; i.e., the natural numbers.

While potential infinity is an essential component of any form of mathematics, may it be reducible to arithmetic or geometry², the step needed to account for actual infinity is one that poses theoretical difficulties. In which sense an infinite totality should be understood as one complete thing? The basic distinction between one and many seems to blur when infinity is reached. Moreover, the practice-oriented origin of mathematics cannot offer examples to fill the conceptual gap needed to pass from potential to actual infinity.

Although a complete understanding of the Aristotelian position is still a matter of debate ([14], [16], [7]), the influence of Aristotelian philosophy had the effect of banning actual infinity from mathematics, being considered out of reach for human rationality. This cultural background influenced the subsequent two thousand year of history, therefore obstructing a mathematical treatment of actual infinity.

Besides rare exceptions, like Saint Augustin and Leibniz, actual infinity did not receive many words of appraisal and the existence of actually infinite mathematical collections was considered contradictory or simply meaningless. Because infinity was a common attribute of God or Its properties, the human rational finite character was considered an insurmountable limit in understanding infinity.

Still in 1831 Gauss wrote, in a letter to Heinrich Schumacher, “[...] I protest above all against the use of an infinite quantity as a *completed* one, which in mathematics is never allowed. The infinite is only a *faon de parler*, in which one properly speaks of limits.” These words exemplify a common attitude that Kant had acknowledged few years before in his *Critic of the Pure Reason*, placing infinity among the antinomies of reason³.

²Not only the size of numbers is not bounded, but neither are the length of segments in Euclid’s geometry. So to make possible arbitrary large constructions.

³The role of Kant is, however, not only negative. An important change that is present in his work, and that will later help the development of modern logic and indirectly a more liberal treatment of infinity, is the idea that existence is not a property of objects. Indeed, from this perspective existential statements do not fit

Gauss's need to talk so authoritatively against infinity is explained by a cultural shift that prepared the ground for its triumph in the late XIX century. Although still in a potential form, infinity entered the scene of mathematics in the modern era, when the infinitesimal analysis and the related notion of limit became essential tools in the application of mathematics to nature.

It was with the urge to better grasp such a fruitful tool that mathematicians of the XIX century dedicated their efforts in what is now called the foundation of analysis, aimed at freeing calculus from its geometrical justification. These studies brought not only the ϵ - δ presentation of calculus, but also a deeper and theoretically mature reflection on the nature and definition of the real line.

It is in this mathematical context that Cantor obtained his first fundamental results. The act of birth of a theory of actual infinity is the publication of *On a property of the set of real algebraic numbers* [2], in 1874. In this paper, although with not much ado, Cantor proved what is nowadays called Cantor's theorem: the cardinality of \mathbb{R} is strictly greater than that of \mathbb{N} .

2. CANTOR'S THEORY OF INFINITY

The form in which Cantor's theorem first appeared in print, in 1874, it is not with the well-known diagonal argument that will become the characteristic mark of his logical contribution to the theory of infinity. Instead, Cantor made use of the Bolzano-Weierstrass theorem, showing that for any enumeration of real numbers indexed by natural numbers, it is possible to find a new real number that does not belong to the enumeration. Indeed, given a real interval (a, b) , either only finitely many numbers of the enumeration fall within the interval, and therefore the theorem follows, or infinitely many do. In the latter case these numbers determine a countable family of closed nested intervals to which we can apply Bolzano-Weierstrass theorem.

2.1. The concept of power. The key ingredient for the modern formulation of Cantor's theorem is the notion of power, that is, of cardinality. And it is only in 1878, in the paper *A contribution to the theory of manifolds* [3], that Cantor defines the notion of "having the same number of elements" in terms of a one-one correspondence. More

anymore in the Aristotelian structure of subject-predicate. A thorough analysis of existential statements will lead Frege to introduce a distinction between quantifiers and other logical constants, while others to offer different views on the relationships between existential sentences and mathematical reality. We will come back to this in §3.

formally, two sets A and B have the same cardinality or power in case there is a bijective function $f : A \rightarrow B$; in symbols $|A| = |B|$.

Although quite natural for our modern look, this move was overcoming a long tradition of criticism to infinity due to the counterintuitive consequences of witnessing sameness in cardinality by bijective functions. In 1638, Galileo in the *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* argued that ‘more’ or ‘less’ could not be applied to infinite quantities, since there were as many squares of natural numbers as natural numbers; and this contrasted with the fact that squares form a proper subset of natural numbers.

Therefore, Cantor posed at the very heart of his theory of infinity a notion of cardinality that considered Galileo’s observation as non-paradoxical. Where Cantor lead many followed. Frege will then base the definition of natural numbers on the concept of “being in a bijection with”, in [10] §63, and Dedekind, in [8] §5, will propose a definition of infinity in terms of being in bijection with a proper subset. Even more interestingly, Dedekind defined the notion of finite⁴ in terms of not being infinite in the above sense, therefore inverting the conceptual order of priority between finite and infinite.

This notion of cardinality, although based on the acceptance of the counterintuitive properties of the infinite, was shown to have properties that nicely fit with its intuitive interpretation.

First of all it is possible to define a notion of order between cardinalities, saying that $|A| \leq |B|$ iff there is an injective function from A to B (and $|A| < |B|$ iff $|A| \leq |B|$ and there is no bijective functions from A to B), in such a way that the obvious properties of a relation between magnitudes are respected. Not only this \leq -relation is clearly reflexive and transitive, but it is an important result, bearing the names of Schröder and Bernstein, that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$, thus showing the antisymmetry of the \leq -relation.

Moreover Cantor showed that cardinalities could generalize the operations of finite quantities, since sum, product, and exponentiation could be defined as follows:

$$\begin{aligned} |X| + |Y| &= |X \times \{0\} \cup Y \times \{1\}|, \\ |X| \cdot |Y| &= |X \times Y|, \\ |X|^{|Y|} &= |X^Y| \text{ where } X^Y = \{f : f \text{ is a function from } Y \text{ to } X\}. \end{aligned}$$

These operations do generalize to arbitrary cardinalities the usual arithmetical operations on natural numbers, since if X has n -elements

⁴This is what is nowadays called Dadekind-finite. This notion is equivalent to the standard one in the presence of the Axiom of Choice, but in the absence it is not necessarily so.

and Y m -elements, we get that $X \times Y$ has $n \cdot m$ -many elements, Y^X the set of functions $f : X \rightarrow Y$ has m^n -many elements, and $X \times \{0\} \cup Y \times \{1\}$ has $m + n$ -many elements.

In other terms, the notion of cardinality was shown to be *materially adequate* to play the role of measuring size, while extending the usual notion of finite quantity.

2.2. The Continuum Problem. With the notion of power at disposal it is possible to express more precisely a concern that Cantor already discussed at the end of his 1874 paper and that will then be called the Continuum Problem. Since $|\mathbb{N}| < |\mathbb{R}|$, is it possible to display other infinite cardinalities among the infinite subsets of \mathbb{R} ? Cantor thought that that was not the case. This proposition was then called the Continuum Hypothesis (CH) and can be formalized saying that if $X \subseteq \mathbb{R}$, is infinite, then either $|X| = |\mathbb{R}|$, or $|X| = |\mathbb{N}|$.

Not only Cantor spent much of his life in the attempt to prove (or sometimes disprove) CH, but much of the subsequent history of set theory was driven by the attempt to formally decide whether CH holds. The strategies to attack this difficult problem can be classified into two general categories, examples of which can be found already in the work of Cantor. It is possible either to study the properties of different subsets of the reals of growing complexity, or to develop a general theory of infinite cardinalities, with the aim of finding the right place of $|\mathbb{R}|$ and $|\mathbb{N}|$ within this theory. The former gave rise to *descriptive set theory*, while the latter to *pure set theory*.

The result that later will be considered the first step towards descriptive set theory was obtained by Cantor and Bendixson, who showed that any closed subset of \mathbb{R} has the perfect set property (PSP): i.e., either it is countable or it is a perfect set, that is, closed with no isolated point. Then, since a perfect set is always in bijection with \mathbb{R} , the Cantor-Bendixson showed that closed sets could not offer a counterexample to CH. The study of regularity properties as PSP will later include Lebesgue measurability and the Baire property and through the contributions of the French and the Russian schools⁵, at the beginning of the XX century, will create the main topic of investigation of descriptive set theory, showing that larger and larger classes cannot offer counterexamples to CH⁶.

⁵An interesting presentation of this story can be found in [12], where the development of set theory in Russia is also linked to a mystical orthodox heresy, the *Name Worshipping*, that recognized the creative power of God's name.

⁶That Borel sets have the PSP was later shown by Martin in ZFC [20], while for larger classes large cardinals are needed. Moreover a fruitful interaction between

For what concerns pure set theory, Cantor investigated other common mathematical structures in order to find new examples of infinite cardinalities. His efforts were, nonetheless, unsuccessful and surprising at the same time. One unexpected result, published by Cantor in the 1878 paper, was that \mathbb{R} and \mathbb{R}^n have the same cardinality. But the real turning point in pure set theory happened when Cantor placed the notion of ordinal number at the very center of his theory of infinity. Indeed, the theory of well-ordered sets offered concrete examples of new infinite sets, which could not otherwise been found in ordinary mathematical practice.

2.3. The notion of ordinal. A linear order $(X, <)$ is a well order if $<$ is a linear order on X such that any non-empty subset of X has a least element according to $<$. For example, the natural numbers with the usual order (i.e. the structure $(\mathbb{N}, <)$) is a well order.

Clearly an initial segment of \mathbb{N} , say $\{0, 1, 2, \dots, n-1\}$, is a finite well-ordered set. Then, let us call \mathbf{n} the well-ordered set $\{0, 1, 2, \dots, n-1\}$ (i.e. the set $\{0, 1, 2, \dots, n-1\}$ together with the natural well order of its elements) and notice that $\mathbf{n} \cup \{n\} = \{0, 1, 2, \dots, n\}$. Consequently, we can identify this simple set theoretic union with the successor operator, as defined in \mathbb{N} , and thus give to $\mathbf{n} \cup \{n\}$ the name $\mathbf{n} + \mathbf{1}$. Since, for every $n \in \mathbb{N}$, we have $\mathbf{n} \in \mathbf{n} + \mathbf{1}$ and since \mathbf{n} represents an ordered set on length n , we can collect all \mathbf{n} 's together and call this new well-ordered set ω . Therefore, ω exemplifies the well order of $\mathbb{N} = \{0, 1, 2, \dots\}$. But now, having at disposal a set theoretic version of the successor operator we can define $\omega + \mathbf{1}$ as $\omega \cup \{\omega\}$ and then continue to create longer and longer well-ordered sets.

The well-ordered sets thus constructed are called *ordinals* and they form a collection of canonical representatives of well-ordered sets. Moreover, while ω and $\omega + \mathbf{1}$, as ordered sets, display two different types of orders, nonetheless they have the same cardinality: $|\mathbb{N}| = |\omega| = |\omega + \mathbf{1}|$. What Cantor surprisingly discovered is that collecting all possible ordinals of, at most, countable cardinality we can naturally order these well-ordered sets confronting their lengths (e.g. ω comes after any \mathbf{n} , but before $\omega + \mathbf{1}$). The resulting ordered set, called ω_1 , however has not countable cardinality but any of its initial segments is countable. Thus ω_1 was defined to be the least uncountable ordinal, in this natural order induced by the length of ordinals.

descriptive and pure set theory allowed to isolate an important class of sets, called *universally Baire sets*, that plays a fundamental role in one of the most promising programs towards the solution of CH, created and pursued by Woodin [22].

The above argument can of course be iterated. Collecting together all ordinals with the same cardinality of ω_1 we can create a new ordinal of higher power, called ω_2 , that is the second infinite cardinal after $|\mathbb{N}|$. Ordinals thus offer both concrete examples of larger and larger cardinalities and a way to enumerate them, by what Cantor called the \aleph -function. The \aleph -function is a function that uses ordinals to well order infinite cardinalities: $\aleph(0) = |\mathbb{N}|$, then called \aleph_0 , $\aleph(1) = |\omega_1|$, called \aleph_1 , $\aleph(2) = |\omega_2|$ called \aleph_2 , and so on. In this sense ordinals allowed Cantor to reach larger and larger examples of infinite powers; ones he could not find by analyzing well-known structures taken from mathematical practice.

Interestingly, also the first approach to the solution of the Continuum Problem makes an extensive use of ordinals. Indeed the definition of more and more complex subsets of \mathbb{R} like Borel sets, analytic sets (i.e. projection of closed subsets of \mathbb{R}^2), co-analytic sets (i.e. complements of analytic sets), and projective sets (obtained by the operation of projection and complementation from subsets of \mathbb{R}^n , for $n \in \mathbb{N}$) is given by the possibility to iterate beyond the finite case the basic set theoretic operations.

In other terms, the key ingredient that allowed Cantor to develop a concrete and well-determined theory of infinity was the notion of *ordinal number*. Cantor viewed this notion as a natural extension of the concept of number through the transfinite. In 1883 Cantor attempted a conceptually broader presentation of his ideas in the paper *Foundations of a general theory of manifolds: a mathematico-philosophical investigation into the theory of the infinite* [5], justifying the introduction of the ordinals from the following perspective.

I am so dependent on this extension of the number concept that without it I should be unable to take the smallest step forward in the theory of sets; this circumstance is the justification (or, if need be, the apology) for the fact that I introduce seemingly exotic ideas into my work. For what is at stake is the extension or continuation of the sequence of integers into the infinite; and daring though this step may seem, I can nevertheless express, not only the hope, but the firm conviction that with time this extension will have to be regarded as thoroughly simple, proper, and natural⁷.

⁷[5], p. 882.

As in the case of the notion of power, Cantor was firmly convinced that ordinals displayed properties analogous to those of the finite natural numbers. This attitude that has been labelled *Cantor's finitism* by Hallett [13] was used not only to justify the extension of the concept of number, but also as a guiding principle for the introduction of the rules that governed the infinite, in analogy with the finite. Indeed, Cantor's finitism can be easily explained as the conviction of a substantial uniformity between the finite and infinite realms.

On this ground Cantor extended the arithmetical operations of sum and product to well ordered sets.

$(\alpha, <_\alpha) + (\beta, <_\beta) = (\alpha \cup \beta, <_{\alpha+\beta})$, where $<_{\alpha+\beta}$ is the order that enumerates first all elements of α , according to $<_\alpha$, and only later all elements of β , according to $<_\beta$.

$(\alpha, <_\alpha) \cdot (\beta, <_\beta)$ is given by the ordered set obtained by laying down an ordered series of copies of $(\alpha, <_\alpha)$, according to the order displayed by $(\beta, <_\beta)$. Therefore the order of the product is given by the lexicographical order in which the first component, from $(\beta, <_\beta)$, tells in which copy of $(\alpha, <_\alpha)$ we are, while the second where we are within that copy.

The application of these operations to ordinal numbers allowed Cantor to develop a rich and interesting ordinal arithmetic and to counter old objections to the possibility to reason with infinite quantities. A common objection to infinity, that dated back to Aristotle, consisted in saying that an infinite quantity would annihilate the addition of any finite quantity, thus making impossible any such arithmetic. But Cantor showed that although this is the case for $\mathbf{1} + \omega = \omega$, since ω was defined as the collection of all \mathbf{n} 's, this is not the case for $\omega + \mathbf{1} \neq \omega$.

This example also shows a delicate aspect of Cantor's theory of infinity. Although developed in accordance with the rules of arithmetic and as an extension of the concept of number, Cantor's theory displayed also the differences between the finite and the infinite; in the above example the failure of the commutativity law for the ordinal sum.

The analogy with the finite case cannot completely justify the new principles on which Cantor's theory of infinity is based, therefore opening an important debate on the justification of the axioms of set theory that is still alive in nowadays ([1], [18], [17], [15]). Far from being a problem, this remark helps to recognize the breadth of Cantor's contributions, where mathematical and philosophical elements are deeply intertwined.

2.4. Types of infinity. An important novelty in Cantor's treatment of infinity was not only the creation of a new arithmetic for the infinite

numbers, but a new classification of kinds of infinities. The difference between finite and infinite was made smaller, but a form of impenetrable infinity remained.

Cantor distinguished between the *transfinite*, where we find the infinite quantities like ordinals, that although infinite can still be augmented, and the *Absolute*, that is the infinite in its proper form and that cannot be subject of any human inquiry.

I have no doubt that, as we pursue this path ever further [i.e. the study of larger and larger infinite numbers], we shall never reach a boundary that cannot be crossed; but that we shall also never achieve even an approximate conception of the absolute. The absolute can only be acknowledged but never known—and not even approximately known. [...] The absolutely infinite sequence of numbers thus seems to me to be an appropriate symbol of the absolute [...]⁸.

In other terms, although the possibility of generating larger and larger infinite numbers will never come to an end, there is still a notion of infinity that cannot be attained by this process, one that Cantor identified with the whole collection of infinite numbers.

The description of the Absolute took in Cantor's writing a theological character that cannot be separated by his work, at pain of losing unity and coherence of his theory of infinity. Indeed, not only Cantor viewed his work as a sort of revelation, but the main existence of infinite numbers was justified in theological term, recurring to the conviction that any possibility is actually realized in God's mind.

In more mathematical terms, it is easy to show that the collection of all transfinite numbers cannot have a cardinality. In some sense, it is too big to be analyzed in terms of cardinality. This was a fact well known to Cantor and, far from being paradoxical, it was considered as a sign of the incommensurability of the Absolute and of the distance between men and God.

But if the collection of all transfinite numbers was out of reach for human understanding, on what ground Cantor acknowledged the existence of transfinite numbers? Again the key notion is that of ordinal number, since the notion of power is realized in terms of ordinals; i.e. \aleph_0 is exemplified by ω , \aleph_1 by ω_1 , and so on.

⁸[5], p. 916.

3. LANGUAGE AND MATHEMATICAL REALITY

Initially, ordinals formed a mere notational system and were presented as *symbols of infinity* [4]. They indexed the steps needed to define new sets of reals, thus allowing to iterate these constructions into the transfinite: i.e. beyond ω . The first example of such operation was the definition of a *derived set* i.e., the set of all limit points of a given $X \subseteq \mathbb{R}$. Indeed, the collection of all accumulation points of a set can itself have accumulation points and Cantor discovered that the process of taking the derived set did not necessarily end after ω -many steps.

But the shift from indexes to full existent mathematical objects is not innocuous and needed to be justified mathematically and philosophically. For what concerns the former, Cantor appealed to what has been later called the *domain principle*. This consists in arguing that if a quantity can take different values, then the domain of variability must itself exist. Therefore, for example the set of natural numbers must be an existent complete totality, since we use variables for natural numbers that can range over the entire set. The domain principle is simply a collapse of the notions of potential and actual infinity—in favor of the actual infinity—and thus it is a *petitio principii* more than an argument. However, in Cantor’s view the domain principle received a clear and strong justification on a theological ground, since there is no difference between potentiality and actuality in God’s eyes. Of course this cannot help a reader that rejects an argument based on essential properties of the divine intellect. However, deprived of this theological character, Cantor’s position is a simple declaration of realism with respect to mathematical objects.

In his more philosophical defense of his ordinal theory, Cantor [5] exposed with more details his view on existence in mathematics. Cantor distinguished among two senses in which natural numbers exist, calling them *immanent reality* and *transient reality*. This is how these two forms of existence are described, in Cantor’s words.

First, we may regard the integers as actual in so far as, on the basis of definitions, they occupy an entirely determinate place in our understanding, are well distinguished from all other parts of our thought, and stand to them in determinate relationships, and thus modify the substance of our mind in a determinate way; let us call this kind of reality of our numbers their *intrasubjective* or *immanent* reality. But then, reality can also be ascribed to numbers to the extent that they must be taken

as an expression or copy of the events and relationships in the external world which confronts the intellect, or to the extent that, for instance, the various number-classes (I), (II), (III), etc. are representatives of powers that actually occur in physical and mental nature. I call this second kind of reality the *transsubjective* or the *transient* reality of the integers⁹.

In other words, mathematical objects exist in so far as we can define them, but also in the strongest sense of the word existence. Moreover, Cantor sees a perfect correspondence between these two forms of reality, to the extent that mathematicians should only care about the immanent reality since the “linking of both realities has its true foundation in the *unity* of the *all to which we ourselves belong*.” [5], p. 896. This is the theoretical ground on which Cantor affirmed the *freedom* of mathematics. The mathematical work is completely free, since what seems to be the creation of new ideas, concepts, and objects by means of definitions, it is in essence the discovery of an autonomous already existing reality.

3.1. From naïve to formal. The deep connection between language and mathematical reality displayed in Cantor’s work is a common trait of many authors of the end of the XIX century. Remember that Dedekind in his famous paper on the foundations of number theory, *Was sind und was sollen die Zahlen?* [8], summarizes his work as follows: “My answer to the problem propounded in the title of this paper is, then, briefly this: number are free creation of human mind”. Dedekind’s work is often considered closer to what will later become a logicist position in the foundations of mathematics, quite distant from Cantor’s realism. But to appreciate the transversality of this tight connection between language and mathematical reality, remember that also Hilbert expressed a similar belief when he wrote, in a letter to Frege [11], in 1899, that his criterion for existence and truth in mathematics was consistency; a notion that he saw from a syntactic, hence linguistic, point of view.

One might conjecture that the emergence of the philosophical difficulties of the connection between language and mathematical reality, and therefore of the many different perspectives cited above, is inextricably connected with the emergence of infinity as a truly mathematical concept. And it might not be the case that the study of a notion so

⁹[5], p. 895-896.

divergent from our everyday experience happened in a time when geometry started to abandon the intuitive Euclidian interpretation and embraced a more liberal perspective. In [9] we find an interesting reconstruction of the *fil rouge* that connected the works of Riemann, Dedekind, and Cantor and that well explains the initial terminology of manifold, to refer to sets, and of the theory of manifolds, to refer to set theory.

Be as it may, it was an uncritical perspective of the relationships between language and mathematical reality that motivated the tacit adoption of what will later be called the *principle of naïve comprehension*: the possibility to determine a set by a property. Although Frege was a fierce opponent of the creative power of language, saying that “This theory imagines that all we need do is make postulates; that these are satisfied then goes without saying. It conduct itself like a God, who can create by his mere word whatever he wants” [10], his work too fell victim of this uncritical perspective, when Russell’s paradox, in 1901, showed that it is possible to define objects that do not exist.

Russell’s famous paradox states that the set of all objects that do not belong to themselves—which is itself a well defined property—cannot exist. This is easily shown by the following argument. Define $R = \{x : x \notin x\}$ and notice that if R were a set, we could ask whether $R \in R$. If this was the case R would satisfy its defining property, i.e. it would be a set x which does not belong to itself, hence $R \notin R$. By a similar argument we can also infer that if $R \notin R$, then $R \in R$. Thus we get that $R \in R$ if and only if $R \notin R$: a contradiction.

But Cantor’s faith in a perfect correspondence between an immanent and a transient reality was not only disproved by Russell’s paradox, but also by Zermelo’s proof of the Well-Ordering Theorem (WOT) in 1904 [23]. Although Cantor went far enough to say that the possibility of well-ordering a set needed to be considered as a law of thought, nonetheless the intense debate that followed Zermelo’s introduction of the Axiom of Choice (AC) in the proof the WOT made clear that the non-constructive character of AC was responsible of the existence of sets lacking a linguistic characterization; e.g. choice functions to which it does not correspond any law. And this too is a failure of a naïve correspondence between language and mathematical reality. Indeed it showed that there are objects that cannot be named.

After these events it become clear that this uncritical perspective towards the descriptive or creative role of language was problematic.

Cantor's theory of sets needed a more secure ground: the linguistic access to infinity needed to be tamed and justified. This is why Hilbert—one of the strongest defender of infinity in mathematics—suggested Zermelo to apply his new version of the *axiomatic method* to set theory.

In Zermelo’s words, this task amounted to the following.

Now in the present paper I intend to show how the entire theory created by Cantor and Dedekind can be reduced to a few definitions and seven principles, or axioms, which appear to be mutually independent¹⁰.

It is therefore in 1908 that starts the history of axiomatic set theory, one in which the notion of set and its axiomatic treatment will offer the tool to secure the fast growing free mathematics, that will later be called, by Hilbert, “the Paradise the Cantor created for us”. But this is another story.

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