On the naturalness of new axioms in set theory

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Abstract

In this article we analyze the notion of natural axiom in set theory. To this aim we review the intrinsic-extrinsic dichotomy, finding both theoretical and practical difficulties in its use. We will describe and discuss a theoretical framework, that we will call conceptual realism, where the standard justification strategy is usually placed. In outlining our view, we suggest that the extensive use of naturalness calls for a revision of the standard strategy, in favor of a justification that takes into account also the historical process that lead to the formalization of set theory. Specifically we will argue that an axiom can be considered natural when it helps the clarification of the notion of arbitrary set.

Introduction

The foundations of set theory is one of the most exciting areas of research in the field of formal sciences that combines both challenging mathematical problems and a deep philosophical reflection. Since the discovery of the phenomenon of incompleteness by Gödel – and even more after the invention of forcing by Cohen – it became clear that if set theory was to be considered the right foundations for mathematics the widespread presence of independence results had to be contained. As a consequence, in his famous article on the Continuum Problem ([Gödel, 1983]) Gödel suggested what is now called *Gödel's program* for new axioms in set theory, that was later refined by Woodin in his programmatic paper on the Continuum Hypothesis ([Woodin, 2001]). The background motivation of both programs consists in interpreting the limits of ZFC not as intrinsic to set theory, but only as a defect of the formal presentation that Zermelo and Fraenkel – among others – gave to Cantor's theory of sets. Hence Gödel's program proposes to supplement ZFC with new axioms able to give a definitive solutions to independent problems and, so, to restore the foundational role of set theory.

> [T]hese axioms [i.e. large cardinal axioms] show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far[1].

Philosophically, this move had the consequence of opening a major discussion on the foundations of set theory and the justification of new axioms extending ZFC.

In what follows we will try to understand what 'without arbitrariness' means and how the notion of naturalness acts in the clarification of this expression. It is indeed a matter of fact that the reference to natural components of mathematics, in general, and more specifically the attribution of naturalness to good axiom candidates has become fairly common. Our aim consists in unveiling the philosophical background of mathematical reasons. Indeed we believe that, due to the technical character of this subject, philosophical ideas are often obscured by mathematical results. In doing so we will critically discuss two major difficulties of the justification of new axioms: first the belief that the use of the intrinsic-extrinsic dichotomy helps in a philosophical elucidation of the problem, second the assumption that it is with respect to a stable and well defined concept of set that we should develop Gödel's, or Woodin's, program: a step by step solution of all set-theoretical problems.

The article is structured as follows. In section 1 we clear the philosophical background where to pursuit a meaningful discussion of the justification of new axioms. Under the name conceptual realism we will then identify the theoretical assumptions implicit in what we consider to be the standard[2] strategy of justification of new set-theoretical principles. In the attempt of clarifying the notion of natural axiom, in section 2, we review the distinction between intrinsic and extrinsic reasons, and in section 3 we describe both theoretical and practical difficulties in the use of this dichotomy. Then, in section 4 we discuss the problem, connected with conceptual realism, of a global perspective in the justification of axioms meant to pursuit Gödel's program. Finally, in section 5 we will suggest a different strategy of justification: one that takes into account the theoretical and philosophical reasons that motivate the axiomatization of a theory. We will then explain in which sense we believe that an axiom should be considered natural and we will find in the notion of arbitrary set the idea with respect to which is possible to argue in favor of the naturalness of new axioms

[1] [Gödel, 1983], p. 182, in [Gödel, 1990].

[2] The term 'standard' refers here to what is common among set-theorists working in the context of Woodin's program. In the next sections we will clarify more precisely the theoretical meaning of standard.

of set theory. In discussing the reason for the attribution of naturalness we will describe a new theoretical framework that will try to overcome both the limits of the intrinsic-extrinsic dichotomy and the difficulties that conceptual realism encounters in the justification of new local axioms of set theory.

1 On the justification of axioms

One of the major imports of modern axiomatics is a radical change in perspective about truth and meaning in mathematics. We assisted at a progressive detachment from the ancient practice of considering the basic principles of a theory as sentences expressing self-evident propositions, towards a more abstract conception that views axioms as legitimate components of mathematical enquire.

The old notion rested on a conception of truth by reference that considered the direct link with the subject matter of a theory as a secure ground for both validity and justification. This point of view is exemplified by the classical idea that the truth of the axioms rests ultimately on their ability to capture essential properties of the objects described by a theory. Against this attitude we find a modern perspective that ceases to consider the truth as an evident property of axiom, but instead as a notion that depends on a complex mixture of internal or external mathematical reasons. As an extreme example of this perspective we may find Hilbert's idea according to which truth is considered an internal property of an axiomatic system and the ultimate criterion for truth is thus consistency.

> As long as I have been thinking, writing and lecturing on these things, I have been saying the exact reverse: if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by them exist. This is for me the criterion of truth and existence[3].

Although Hilbert thought to have eliminated the problem of justification by means of a philosophy of mathematics based on the extensive use of implicit definitions, nevertheless the widespread presence of independence in set theory urged a choice between incompatible axiom candidates and consequently the justification of our preferences. However, how to conciliate a modern perspective on axioms and the need to extend ZFC without arbitrariness?

Of course truth remains the main concern of mathematical research, but it is clear that it is not anymore an intuitive or evident property. Moreover, the

[3]Letter from Hilbert to Frege December 29th, 1899; in [Frege, 1980], pp. 39-40.

idea that new advances in set theory comes from the mutual interaction between conceptual analysis and the confrontation of our expectations with the outcomes of our discoveries resembles closely a scientific method applied to pure mathematics. Therefore, in order to avoid the dependency of the justification of new axioms on an alleged theory of truth, we might assume a minimal conception of truth, viewing it as a limit notion that, although playing the role of a regulative idea, cannot be considered attained at a given moment of our scientific progress. In other terms, we maintain a sharp separation between the positive outcome of a justification and the truth of an axiom.

Consequently, in dealing with the problem of justification we do not intend to deal directly with the matter of truth in mathematics. Indeed we believe that justification, alone, cannot give the certainty able to ground the truth of an axiom. As part of a dynamic process, the reasons proposed should only be seen as suggestions that point in the direction where to seek truth. Indeed, the main fact that the justification process happens to be revised called for a qualification of the role of truth in this process.

Therefore, it is important to distinguish between the fact that an axiom is true and the reasons – connected with issues of meaning – in virtue of which it is true. Although, from a contemporary perspective, the truth of an axioms is deeply intertwined with mathematical reasons, more philosophical reasons are needed for its justification, precisely because the notion of meaning is involved.

Since the center of our interest is the justification of new axioms, we set the stage of our discussion choosing a non formalist philosophical perspective. As a matter of fact either our problem is philosophically trivial – in a formalist setting it is indeed solved vacuously – or we should then be allowed to assume the presence of a correspondence between syntax and semantics, able to inform the criteria of justification. There are many reasons for such a choice: in first place the difficulties of reductionist formalist positions – like eliminative structuralism[4] – in dealing with foundational theories like set theory, but even more importantly because we believe that on this particular matter a general formalist position should be more substantial than the simple rejection of the problem of justification.

Thus, it is within the context of a discussion on meaning and reference that we can tackle the problem of the justification of new axioms of set theory. But once a correspondentist theory has been deprived of its content of truth, what is left is the link between syntax and semantics given by the meaning

[4]We have in mind here the position that does not admit the existence of either mathematical object, or structures and that proposes to reduce the latter to some more fundamental notion like for example set theory ([Linnebo, 2008]) or modality ([Hellman, 1989]).

of mathematical propositions. Indeed, at a very general level, we may have justified reasons to accept an axiom when we are in the position to argue that it accords with the basic ideas or principles formalized by the theory to which the axiom belongs. The difficult philosophical task here is thus to determine the theoretical status of these ideas or principles. A possible easy solution consists in substituting objects with objective concepts, however, as we will discuss later, this move is problematic in many respects.

Let us now come back to set theory. The problem of justification of new axioms has been shaped by Gödel's program: a step by step supplementation of the axioms of ZFC aimed at finding a solution to concrete mathematical questions like the Continuum Problem i.e., the problem of determining the cardinality of the set of real numbers. In more recent years this program has been taken up by Woodin who proposed a step by step program for the completeness of the first initial segments of the universe of all sets.

Without committing ourselves to a general philosophical description of Gödel's or of Woodin's position, we would like here to give a rough description of the conceptual framework that underlines both programs. It is a common understanding of the problem that meaning points at a – not universally understood – concept of set, able to legitimate directly the axioms of set theory or to play the role of an abstract idea with respect to a more concrete set-theoretical reality. This general picture is supplemented by two general features that shape the related justification strategy.

In first place we find the belief in the existence of a clear and stable notion of set (e. g. the iterative conception of set: the notion of "set of") able to justify new set-theoretical principles. In other words, the belief that it is possible to give meaning to new axioms in terms of fundamental properties of *the* concept of set. Second, the conviction that the notion of set is sufficiently well shaped that the solution of a mathematical problem depends on the recognition of a particular property of this concept. For example, we may believe that it is the notion of "set of" that needs to be analyzed – directly, in a more philosophical way or indirectly in a more mathematical way – in search for a solution to the Continuum Problem.

A consequence of these two ideas is a weak form of realism, since, in order to justify a statement, it is assumed that there is something in virtue of what the justification process can be satisfactory conducted. Since it suggests a correspondence between axioms and concepts, we may call this attitude conceptual realism[5]. Furthermore, a consequence of this attitude is a global point of view

[5]This notion is weaker than Gödel's conceptual realism as described in [Martin, 2005], but as we will see in the discussion of the analytic and synthetic distinction it is connected.

on set theory that links the solution of specific problems to the general concept of set (whatever this may be): only its clarification – via a conceptual analysis or via the understanding of mathematical results – is able to solve a set theoretical problem. Even if the problem is local as the Continuum Problem. Thus the uniformity of the universe of set witnessed by the reflection principles is extended to the conceptual level: it is the general concept of set that determines the solution of (even local) set theoretical questions.

It is with respect to these two ideas that we will try to understand what the use of naturalness suggests in the process of justification. Before that, let us offer another interesting quote, again by Gödel, that connects the idea of extending ZFC without arbitrariness and the notion of natural axiom. We stress that although this notion has been used in the literature after Gödel, for example in [Bagaria, 2004] or in [Friedman, 2006], however it has not been subject of a sufficient theoretical clarification.

> The proposition [V = L]... added as a new axiom, seems to give a *natural* completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way.[6]

As a first attempt in the clarification of the notion of naturalness we propose to understand its place within the classification offered by the intrinsic-extrinsic dichotomy. Indeed, the latter is considered having both a mutually exclusive and a jointly exhaustive character. Although many different criteria for accepting new axioms have been proposed (at least) for the last fifty years, we choose to discuss intrinsic and extrinsic reasons because these criteria are not specific – as for example maximality[7] or fairness[8] or stability[9] – but are *forms* of justification. For this reason the considerations one could make about them should hold in general without been affected by the specific context of their application. Our aim is to unveil the philosophical ideas behind the standard strategy of justification of new axioms sketched above and not to move a criticism towards specific instances of it. Indeed, in the literature it is possible to find good arguments – often mathematically well informed – that sometimes lack an appropriate elucidation of the philosophical ideas that motivate them. When dealing with the reasons in favor of new axioms it is important to recall that philosophy in not only welcome, but necessary due to the main status of *axioms*, whose validity

[6] [Gödel, 1938], p. 557.
[7] See, among others, [Maddy, 1997].
[8] See [Bagaria, 2004] on this respect.
[9] See [Friedman, 2006] on this respect.

cannot be ascertained with rigorous mathematical reasons based on previously accepted principles, if we want them to be *new*.

2 Is naturalness intrinsic or extrinsic?

First of all it is useful to recall the distinction between intrinsic and extrinsic justifications. This dichotomy has been proposed and discussed in Gödel's work and in Maddy's papers. Both authors share a realist position[10] that consists in a clear and proud platonism in the case of Gödel, while a more articulated position in the case of Maddy.[11]

- *intrinsic reasons*: the justification of an axiom originates from the concept of set itself. Axioms are deduced - in a Kantian sense - by *the* concept of set, that is supposed to be knowable and describable in an axiomatic setting. The rational arguments used in this act of justification borrow their legitimacy from the subject matter of the theory. The objectivity of such an argument rests on the stability of a well definite concept of set.
- *extrinsic reasons*: the burden of the justification of an axiom rests on the success of its use as a set-theoretical principle. An axiom is extrinsically justified if its validity is confirmed by many mathematical facts and is able to give new interesting mathematical results. The prediction-confirmation model, typical of the empirical science, finds its place in a purely mathematical context. A more inductive reasoning is used in this act of justification. The objectivity of an argument rests on the possibility to give an explanation of a mathematical phenomenon by means of a mathematical law.

An aspect that is implicitly sustained by both intrinsic and extrinsic reasons is the descriptive character of a justification strategy. Indeed, both intrinsic and extrinsic justifications apply when the principles we intend to justify are able to describe appropriately aspects of the concept of set or, respectively, of set-theoretical reality. For this reason this dichotomy perfectly fits the context of conceptual realism we described before: the realist component is expressed here by the referential component of the descriptive character, while either the

[10] Or at least they shared. In the latest years Maddy's position evolved toward an attempt to overcome the antithesis between realism and anti-realism. See [Maddy, 2011].

[11] Indeed, she started her reflections maintaining a very strong form of existence with respect to sets (even in space and time, as argued in [Maddy, 1990]), for later developing an anti-(first)philosophical naturalism aimed at justifying Woodin's realist view (see [Maddy, 1997] in this respect).

presence of a nature of the concept, or the fact that mathematical reality is describable by its consequences, presupposes a stable conceptual level.

More concretely, intrinsic reasons presuppose the stability of a concept whose objectivity is autonomous from our formal presentation, but that is able to inform, alone, the criteria for accepting new axioms. Indeed, intrinsic reasons are legitimated both by the independent existence of a concept and by the possibility to describe its essential features by means of formal tools.

Moreover, and more interestingly, we believe that also extrinsic reasons presuppose a form of realism: a realism that assimilates mathematics to natural sciences. As a matter of fact the form of arguments that are named extrinsic resemble closely inferences to the best explanation in a purely mathematical setting. The fact that set-theoretic principles have so many desired consequences that force us to accept them, due to their success, can be seen as confirmations of hypotheses by means of experiments. Axioms are confirmed by theorems and not the opposite. Inductive more than deductive arguments are used in the act of an extrinsic justification. Axioms are meant to describe such a stable mathematical reality and their truth rests upon the possibility to mirror independently valid relations between concepts. This form of justification echoes Russell's description of the foundational studies as described in the 1907 lecture *The Regressive Method of Discovering the Premises of Mathematics.*

> But when we push analysis farther, and get to more ultimate premises, the obviousness becomes less, and the analogy with the procedure of other sciences becomes more visible. The various sciences are distinguished in their subject matter, but as regards method, they seem to differ only in the proportions between the three parts of which every science consists, namely (1) the registration of 'facts', which are what I have called empirical premises; (2) the inductive discovery of hypotheses, or logical premises, to fit the facts; (3) the deduction of new propositions from facts and hypotheses[12].

This form of empiricism in the context of mathematics is strongly linked with the conceptual realism we described before. An analogy is indeed assumed to hold between mathematical reality and physical reality. Axioms are then intended to describe mathematical phenomena, as laws do in the case of nature, and in both cases mathematical, respectively, physical concepts are intended to act as a bridge between "reality" and its formalization.

Now, where to place the naturalness with respect to the division between concepts and reality suggested by the intrinsic-extrinsic dichotomy? Our sug-

[12] [Russell, 1907], p. 282.

gestive answer is that the naturalness of an axiom is sustained both by intrinsic and extrinsic reasons; and this is a hint both of the weakness of this dichotomy and of the peculiarity of the notion of naturalness. Indeed the explicit reference to nature in the attribution of a natural character to a piece of mathematics may here refer to the nature of the concept of set, in the case of intrinsic reasons, or, in the case of extrinsic reasons, to a sufficiently stable set-theoretical reality that, like nature, we may be able to describe in an inductive way with a kind of reasoning that mimics an inference to the best explanation.

The attribution of naturalness to an axiom candidate seems to stress the acceptance of a realist perspective and the presence of a correspondence with a semantical level, whose essential aspects are described axiomatically. However, our discussion of naturalness will intend to convey that this descriptive aspect is only apparent. Indeed, we will argue that the attribution of naturalness is meant to be a prescriptive move of the mathematical work. As a matter of fact, an attentive analysis of the use of the term 'natural' in mathematical practice suggests that, even in keeping with a realist perspective, the justification of an axiom needs a dynamic framework, that the static conceptual realism cannot accommodate.

Before offering our view on naturalness, it is instructive to discuss some difficulties of the standard strategy of justification. Now 'standard' may be understood more theoretically, and precisely, as a justification that relies on conceptual realism and that makes use of the intrinsic-extrinsic dichotomy to classify the arguments in favor of new axioms candidates.

The limits of the standard strategy we discuss will help us showing the need of a new, less theoretically loaded, framework able to take into account more practical considerations, coming from the historical development of set theory. Indeed, in agreement with the modern perspective sketched in Section 1, we consider the process of justification of new axioms as an integral part of mathematical research.

However, it is important to clarify since now that we do not propose naturalness as a new criterion for the justification of new axioms, but as a linguistic indicator that points towards the revision of the standard strategy. As a matter of fact, the possibility to consider the naturalness of an axiom as an intrinsic and as an extrinsic justification is not meant to show that naturalness is a new form of justification, but that the intrinsic-extrinsic dichotomy fails in giving a ready-to-hand classification of the justification criteria. The reasons being, as we will see in the next section, that, on the one hand, the aforementioned dichotomy fails to be a dichotomy and, on the other hand, that it presupposes the outcome of the justification process it means to classify.

In showing the limits of the standard justification we will also discuss critically the basic assumption on which conceptual realism is based: the presence of a well defined concept of set, able to inform directly the criteria of justification. This latter criticism will not only affect a justification strategy based on intrinsic reasons, but the whole framework of conceptual realism. This is because the best understanding of extrinsic criteria we can offer relay on an analogy with natural science that sees in the regularity of mathematical phenomena the clue of the homogeneity of the underlying mathematical reality. Therefore the more or less emphasis of the conceptual component of set theory is only related to the more intensional or extensional perspective on the nature of set theory. In other words, although intrinsic reasons rely directly on the concept of set, we believe that also extrinsic reason deals, now indirectly, with the concept of set, by means of its extension. Indeed the stable properties observed in the everyday mathematical work are considered indicating the presence of a coherent and well-determined organization of mathematical reality, that therefore can be described abstractly by means of an objective concept of set, exactly as a physical model is supposed to describe reality.

We can now proceed in presenting two main difficulties of the standard strategy that we will name dogmas, since they can be seen to found the standard strategy, while being, in our opinion, unfounded.

3 First dogma

As anticipated we now proceed in showing the main difficulties of the standard strategy of justification. We start discussing the limits of the intrinsic-extrinsic dichotomy, epitomizing them in the following principle.

Fact 3.1. First dogma: *not only it is possible to give a clear distinction between intrinsic and extrinsic reasons and to apply them meaningfully in every situation, but moreover this dichotomy adds philosophical clarification to the process of justification of new axioms.*

We believe that a justification strategy solely relaying on intrinsic or extrinsic reasons, although widely used, is problematic in many respects, both from a theoretical and a practical point of view.

3.1 Theoretical difficulties

Let us start by trying to elucidate the following question: when can we say that an axiom is intrinsically justified? Since the source of such arguments is to be found in the nature of the concept of set, or at least in a uniform behavior of mathematical phenomena, we need to argue in favor of a link between a syntactic entity (an axiom) and a conceptual entity (the concept of set). Such a relation is in principle very difficult to ascertain, due to the general problem of finding a safe and faithful bridge between the formal and the informal sides of mathematics. Moreover, in the particular case of set theory, this task is even more difficult. Indeed the very nature of the concept of set is open to a never-ending sequence of specifications that can be hardly captured even at the level of second order logic. These problems apart, how it is possible to match axioms and concepts?

Vague as it may be, we may assume that axioms have conceptual contents, or in other words that they express propositions able to faithful describe some relevant aspects of the concept of set. But on which basis can we say that an axiom captures aspects of the notion of set? Without appealing to the opaque notion of Gödel's intuition – and for which Frege's old warning is apt: "We are all too ready to invoke inner intuition, whenever we cannot produce any other ground of knowledge"[13] – an interesting possible answer can be found in Boolos' famous article on the iterative concept of set ([Boolos, 1971]).

> It seems probable, nevertheless, that whatever justification for accepting the axiom of extensionality there may be, it is more likely to resemble the justification for accepting most of the classical examples of *analytic* sentences, such as "all bachelors are unmarried" or "siblings have siblings" than is the justification for accepting the other axioms of set theory.[14]

The suggestion seems enlightening: when dealing with the intrinsic-extrinsic distinction are not we just proposing again the distinction between analytic and synthetic judgments?

We believe so, since we use similar conceptual tools and argumentative strategies for identifying a reason as intrinsic and a judgment as analytic – respectively, a reason as extrinsic and a judgment as synthetic. Interestingly, this idea has deep roots that can be traced back to Gödel's interpretation of analytic and synthetic, as described in the Gibbs Lecture.

> I wish to repeat that "analytic" here does not mean "true owing to our definitions", but rather "true owing to the nature of the concepts occurring [therein]", in contradistinction to "true owing to the

[13] From §19 of the 1884 edition of [Frege, 1950].

[14] [Boolos, 1971], p. 229.

properties and the behavior of things".[15]

We see here clearly outlined a notion of analytic judgment that refers to the nature of a concept, as intrinsic reasons do, opposed to a notion of synthetic judgment that refers to properties of things, that perfectly corresponds to the idea of justifying an axiom by means of the properties of the concept of set that the latter allows to prove.

At closer inspection the similitude is even more striking. For example, if we are interested in giving reasons for identifying a justification as intrinsic we may appeal to a form of direct link (not necessarily intuition) between an objective mathematical reality and our ability of formalizing it. As a consequence the distinctiveness of the essential features of a mathematical concept (or the reference to independently existent mathematical objects) makes analytic - in the concept of set – the axioms able to capture some fundamental aspects of the concept. Moreover, intrinsic justifications are supported by immediate reasons that recall the criteria for the truth of an analytic statement.

On the contrary, if we maintain that our relation with the concept of set is always mediated by formalization and thus that only formalized set theory is able to shape the concept of set, then any attempt to give intrinsic reasons for believing in an axiom runs into the problems of a circular argument typical of the justification of an analytical statement. Indeed, if axioms are essential for our understanding of the concept of set, then their justification rests ultimately on the axioms themselves.

The analogy between extrinsic reasons and synthetic judgments is even more compelling, since in both cases their justification relays on an external reality – conceptual or concrete – able to express how the meaning of a formal expression (an axiom) relates to an informal domain (the concept of set).

The parallel we propose is meant to show that the use of intrinsic or extrinsic justifications presupposes the knowledge of the meaning of an axiom, or the way this meaning relates to mathematical reality. But this is problematic, since either before the use of these forms of justification we propose a full description of the concept of set and, respectively, an explanation of how meaning is capable of connecting axioms and concept, or any attempt to use intrinsic or extrinsic reasons loses much of its appeal[16]. However, even assuming to have attained such knowledge, once we have a full description of the notion of set, or of how the

[15] [Gödel, 1990], p. 321. The insertions in square brackets are by the editors of Gödel's Collected Works.

[16] We agree we Gödel that this is one of the most difficult task of mathematical logic: "The difficulty is only that we don't perceive the concepts of "concept" and "class" with sufficient distinctiveness, as is shown by the paradoxes." ([Gödel, 1964], p. 140).

meaning of set-theoretical expressions relates to it, we doubt that the process of justification is still non-trivial and of some utility.

The connection between intrinsic and extrinsic justifications and analytic and synthetic judgments is not meant to totally disqualify such dichotomies, but only to stress that these notions should be handled with care. In trying to understand the argumentative strategies that operate in the justification of new axioms in set theory one rapidly gets to some of the most important and difficult problem of the philosophy of mathematics. Indeed, not only Gödel's intuition is a tool in need of philosophical clarification, but also the use of the intrinsic-extrinsic distinction should take into account the sharp criticism that, among others, Quine moved towards the analytic-synthetic dichotomy. As a consequence a careless use of the notions of intrinsic and extrinsic justification in a mathematical context poses more philosophical problems than the ones it helps to solve.

3.2 Practical difficulties

What we argued in the last section remains at a theoretic level. However we believe that also in practice it is not always clear how to apply intrinsic or extrinsic judgments. Let us start with the axioms of ZFC. These axioms are normally taken for granted, once the main focus are *new* axioms. This is of course reasonable and it matches perfectly Hilbert's description of the development of mathematics: the edifice of this science is built without firstly securing its foundations, but one gets back to them only when problems occur. However, as it is possible to see from the next quote, the axioms of ZFC are not always considered as intrinsically justified; quite the opposite.

> I will start with the well-known axioms of Zermelo-Fraenkel set theory, not so much because I [...] have anything particularly new to say about them, but more because I want to counteract the impression that these axioms enjoy a preferred epistemological status not shared by new axiom candidates.[17]

Of course Maddy's naturalism is orthogonal to intrinsic reasons, but next quote, by Boolos, is taken from the same paper where the axiom of extensionality is considered analytic in the concept of set.

> Although they are non derived from the iterative concept, the reason for adopting the axioms of replacement is quite simple: they have

[17] [Maddy, 1988], p. 482.

many desirable consequences and (apparently) no undesirable ones.[18]

Therefore, if it is not a trivial matter to identify the intrinsic axioms even in the case of ZFC, then the property of being intrinsic seems to be a limit notion, more then a concrete property that we can attribute to new set theoretical axioms. As a matter of fact, the main existence of extrinsic reasons is by itself a sign of the difficulty we encounter in assigning intrinsic character to an axiom.

Moreover, the lack of consensus in the application of intrinsic reasons is not balanced by a clear strategy in the application of extrinsic ones. The following example is meant to show how uncertain can be to discern between two apparently equally extrinsically justified set theoretical principles. We will discuss the case of the the Axiom of Choice, that is often considered as the most extrinsically justified axiom among ZFC, and the Axiom of Determinacy, whose fruitful applications represented the success of the first step of Woodin's program.

Definition 3.2. (The Axiom of Determinacy (AD)) *Let* $A \subseteq \omega^\omega$ *(i.e. the set of sequences of natural numbers of length* ω*) and let* $\mathcal{G}_A$ *the game where players I and II choose, in turn, natural numbers*

$$\begin{array}{lccccccc} I & x(0) & & x(2) & & x(4) & & \dots \\ II & & x(1) & & x(3) & & x(5) & \dots \end{array}$$

and that ends after ω*-many steps with the following winning conditions: player I wins when* $x = \langle x(i) : i \in \omega \rangle \in A$ *otherwise player II wins. Then AD is the following statement: for every* $A \subseteq \omega^\omega$*, the game* $\mathcal{G}_A$ *is determined; i.e., there is always a winning strategy*[19] *either for player I, or for player II.*

This axiom that *prima facie* looks very distant from set-theoretic practice has tremendous consequences on many fundamental problems of modern set theory. After the bulk of interesting results that Woodin and others showed to hold in connection with this axiom, AD became a paradigmatic example of an axiom that rests on extrinsic reasons for its acceptance. Nevertheless AD is not compatible with all ZFC axioms. In particular AD implies the negation of the Axiom of Choice (AC), since it implies that all subsets of real numbers are Lebesgue measurable, while by means of AC it is possible to build a non-measurable subset of $\mathbb{R}$. Then how to decide between two apparently extrinsically justified and incompatible axioms?

[18] [Boolos, 1971], p. 229.

[19] A winning strategy is a *function* $\sigma : \mathbb{N}^{<\omega} \to \mathbb{N}$ that tells a player what to play at his or her n-th move, considering what has been played before, and following which that player necessarily wins.

If we look at the subsequent autonomous development of set-theoretical practice we can see that the possibility to have choice functions has been considered unavoidable, and so AD has been considered in need of a reformulation. As a matter of fact set theorists shifted their focus to restricted versions of this axiom; in particular to $\mathrm{AD}^{L(\mathbb{R})}$ that is the Axiom of Determinacy restricted to the inner model constructible from all ordinals and reals. In the presence of large cardinals this structure is a model of ZF axioms, together with the Axiom of Dependent Choice: a principle of choice weaker then AC. Although AD has a lower consistency strength[20] then $\mathrm{AD}^{L(\mathbb{R})}$[21] the latter has been preferred. This retreat from AD to $\mathrm{AD}^{L(\mathbb{R})}$ has not been motivated by clear intrinsic or extrinsic reasons, but by a mixture of considerations of different forms motivated by the goal of accommodating a theory where the most of determinacy could hold together with the most of choice. In other terms, although AC is normally considered as extrinsic in the context of ZFC, when confronted to fruitful axioms that extend ZFC its intrinsic combinatorial value is put forward as a reason for not abandoning the freedom given by choice.

As for the theoretical difficulties we discussed before, here too we face, in practice, the problem of presupposing the outcome of the justification in the application of its criteria; in other words we assume the relevance of some elements from the outset, prior and independently of the criteria of justification we intend to use. Moreover, the case of AD and AC shows clearly that the historical development of set theory may influence the application of the criteria of justification, and consequently that these criteria are not fixed once for all but may vary in accordance to specific cases.

These considerations open the problem of the role of non-mathematical elements – may them coming from history or philosophy – in the process of selection of new axioms. Furthermore we think that a moral we can draw from the interplay between AC and AD is that the intrinsic-extrinsic dichotomy does not pertains to the level of forms of justifications, since these reasons are sensible to the context of their application and so their validity needs to be ascertained case by case. In other words, the discussion of AC and AD shows that the notion of extrinsic reason does not solve, alone, the problem of justification, since we then need other reasons to accept such a justification. But what kind of meta-justifications we are thus looking for?

The difficulties we found in the use of intrinsic and extrinsic reasons are by no

[20] AD is equiconsistent with the existence of infinitely many Woodin cardinals.

[21] Infinitely many Woodin cardinal and a measurable cardinal on top of them are needed for the proof of AD in $L(\mathbb{R})$. In contrast to the case of AD we have here an implication and not a relative consistency dependence. However, to our knowledge, this epistemological difference has never been offered as an argument for the primacy of one of the two principles.

means to be understood neither as the belief that such justifications are useless, nor as the denial that there are cases where this dichotomy can be meaningfully applied. What we discussed here is the wide range of nuances where a mixture of different arguments are needed in order to tip the balance towards one of the two sides. However, we believe that when the justification of new axioms of set theory is subject of a philosophical debate, then these categories loose much of their appeal and it may seem that supporters or opponents of new set-theoretical principles are not gaining much by their use.

4 Second dogma

Before presenting our view on naturalness, it is useful to discuss another aspect of the framework of justification, now in connection with the scope of new axioms. As we hinted in Section 1, besides conceptual realism the standard strategy of justification holds that the general concept of set determines the meaning of new axioms. An example of this attitude can be found in the discussion of the justification of PD (i.e. the Axiom of Determinacy restricted to projective sets).

> Because of their richness and coherence of its consequences, one would like to derive PD itself from more fundamental principles concerning sets in general, principle whose justification is more direct.
>
> We know of one proper extension of ZFC which is as well justified as ZFC itself, namely ZFC + "ZFC is consistent". Extrapolating wildly, we are led to *strong reflection principles*, also known as *large cardinal hypotheses* [...] Reflection principles have some motivation analogous to that of the axioms of ZFC themselves, and indeed the axioms of infinity and replacement of ZFC are equivalent to a reflection schema[22].

This idea then pushes towards the search of global axioms and sustains the implicit idea that new advances in set theory can be achieved only by a further clarification of the general concept. Since we consider this attitude problematic, we will call it, again, a dogma.

Fact 4.1. ***Second dogma****: there is a direct link between the general concept of set and the solution of specific set-theoretical problems; hence it is only with respect to the general concept of set that we may justify a new axiom in set theory.*

[22] In [Martin and Steel, 1989], p. 72.

This dogma has the effect of making problematic the justification of local axioms, because we believe it is not the general concept but a specific instance of it that needs to be the ground of their justification.

Let us argue this point more in details. First of all it is useful to make explicit that we concentrate our analysis on the iterative conception of set, that is normally considered *the* conception of set that is able to determine and justify the axioms of set theory. The reason being that, typically, an argument in favor of an axiom that extends ZFC takes for granted, besides consistency, the existence of characteristic properties of the intended interpretation of ZFC, normally referred to as the cumulative hierarchy V. For example its so-called indescribability, that is the impossibility to give explicit properties, besides the ZFC axioms, that hold for the collection of all sets and not just for an initial segment of V. Indeed, the iterative conception is considered the conceptual underpinning of a cumulative hierarchy structure like V and it is been argued that this conception is able to deduce (in a Kantian sense) most of the ZFC axioms (see [Boolos, 1971]).

These connections between the conceptual and the formal levels can be made precise considering that on the one hand the Levy-Montague Reflection Principle (that is normally considered as expressing the indescribability of V) can be proved in ZFC to be equivalent to the conjunction of the axiom of infinity and the schema of replacement, and that on the other hand Zermelo's quasi-categoricity theorem for ZFC tells us both that the cumulative hierarchy is the right model of these axioms and that even second order logic cannot distinguish between inaccessible levels of the cumulative hierarchy.

It is also useful to specify what we mean by local and global. By global axiom we mean an axiom that deals with all sets in V or at least with an unbounded class of them, while with local axiom one that deals only with sets laying in a proper initial segment of the cumulative hierarchy. In other words global axioms deal with the height of the cumulative hierarchy, while local ones with its width. Therefore, our concerns about the use of the general concept of set for the justification of both local and global axioms can be restated asking: on which ground are we justified in using the same theoretical framework that gave rise to the cumulative hierarchy (i.e. the iterative conception) to argue in favor of axioms that do not influence the hight of the universe of sets, but only its width?

More precisely, if we accept the existence of a stable concept of set and we accept that ZFC axioms correctly (but partially, as shown by Zermelo's theorem) describe it, how can we justify, by means of the same notion, axioms that have consequences only on the lowest levels of the cumulative hierarchy? Although we assume for the sake of the argument that the iterative conception – able to justify ZFC[23] and to tell us what sets are – is so useful in justifying principles expressing the hight of the universe of sets like large cardinal axioms, how can we use arguments linked to the iterative conception for the justification of new local principles able to give a more detailed description of sets laying in an initial segment of the cumulative hierarchy, below the first inaccessible cardinal?

Pushing this point at a more conceptual level we find a related problem. As a matter of fact, even accepting that *the* concept of set can justify the ZFC axioms and the general properties of its intended model(s), on what theoretical ground can we argue that this same notion can determine the notion of "set existing in an initial segment of the cumulative hierarchy" in such a way that the arguments appealing to the former notion can be decisive in the argumentative process of the justification of axioms that deal with the latter notion?

Not only we believe that an argument in favor of a local axiom based on the general concept of set is, without further justification, a deceptive inference, but we also think that such an argument would face the following practical problem. If we call a local concept of set one that is intended to describe sets laying in an initial segment of the cumulative hierarchy, then a local concept should have different specifications than the global one. Indeed, exactly because of the indescribability of V, the main possibility to characterize an initial segment tells us that its axiomatization should be different from that of the universal class. As a consequence, if our aim is to give a (sufficiently) complete description of V by specifying step by step its initial segments, we will eventually face the possibility of outlining properties that are specific to particular sets and not to sets in general – at least the property of laying in an initial segment of V.

An interesting consequence of the rejection of the second dogma is the possibility of giving a substantive answer to what we may call the criticism of vagueness of set theory. By this we mean the following line of argument: the discovery of the widespread presence of independence phenomena in set theory (e.g. the independence of CH) tells us that the concept of set is a vague notion; hence there are good reasons for believing that questions like the Continuum

[23] As Boolos showed in [Boolos, 1971] this justification is far from being straightforward. Moreover we believe that the attempt to justify the axioms of ZFC in terms of the iterative conception does not sufficiently take into account the meaning of Zermelo's theorem. Indeed the idea of a cumulative hierarchy was proposed by Zermelo in his attempt give a consistency proof for ZF. Then we believe that any attempt to justify ZFC axioms in terms of the iterative conception seems distorted if one considers the origin of the latter from the idea of a cumulative hierarchy. However, it is not here the place where to discuss the appropriateness of the justification of the ZFC axioms in terms of the iterative conception. While agreeing on the deep connection between the ZFC axioms, the cumulative hierarchy and the iterative conception, we only want to point out the vacuity of the justification of axioms that can only fix one kind of structure in terms of a conceptual description of that kind of structure.

Problem cannot be settled and, in particular, from the vagueness of the concept of set we may infer that CH has not a well-defined truth value. This argument is perfectly exemplified in [Feferman, 1999] and it has been criticized in [Martin, 2001] with structural considerations, similar to Zermelo's, addressed to the second half of the argument: from vagueness of the concept of set infer the lack of truth-values. On the contrary, if we realize the absence of cogency of the second dogma the argument *à la* Ferferman is blocked at its very beginning. From the independence of CH it is not possible to infer the vagueness of the general notion of set. Although a parallel argument may be then proposed for the notion of countable set, we believe that the appeal to vagueness is, in this context, less persuasive, because on the one hand the ZFC axioms are not meant to formalize the notion of countable set, while on the other hand we have a better understanding of countable sets than we have of sets in general. Indeed, during the last fifty years it has been developed an intense and detailed study of the forcing method (a tool that applies to countable structures), that gave rise to the so-called Forcing Axioms: local axioms able to give a clear picture of the hereditarily countable sets and to decide, among many other things, the cardinality of the Continuum.

These are the reasons why we believe that it is not the general concept of set that can determine the meaning of local axioms, and moreover that can be able to determine the width of the universe of sets and so to be the appropriate theoretical framework for the justification of principles that are meant to pursuit Gödel's or Woodin's programs. It is then not surprising that are not global axioms like large cardinals, but local axioms like Forcing Axioms, that are able to give an answer to the problem of the cardinality of the Continuum.

5 Naturalness revisited

So much for the intrinsic-extrinsic dichotomy and the justifications based on the general concept of set. The criticism we moved to these kinds of justification showed their limits in a philosophical discussion on the acceptance of new axioms. However, we believe that these deficiencies are exemplar on the one hand to understand the use of the notion of naturalness in mathematics and, on the other hand, to suggest a different justification strategy.

In order to elucidate this point we start by widening our analysis and by making explicit our general view of the role and the weight that the notion of naturalness has assumed in contemporary mathematics. We start with the following table that displays the frequency of the use of the terms 'natural' and 'naturalness', between 1940 and 2009, in the texts of the *American Mathematical*

Society database (MathSciNet).

Decade	*Total articles* (T)	*Occurrences* (N)	*Rate* $\left(\frac{N}{T}\right)$
1940 – 1949	40538	602	0.014
1950 – 1959	89158	1935	0.021
1960 – 1969	168567	4802	0.028
1970 – 1979	327427	11500	0.035
1980 – 1989	483143	21026	0.043
1990 – 1999	617522	34032	0.055
2000 – 2009	841470	47056	0.056

Further statistical evidence, in San Mauro and Venturi [2015], confirms that in the last sixty years the reference to natural components in mathematical works increased significantly, without marking an increment of the technical uses of the terms 'natural'– like in expressions as 'natural number', or 'natural transformation'. On the contrary the widespread presence of terms like 'naturally', or of expressions like 'it is natural to see that', clearly manifests a tendency of this notion towards informal components of mathematics.

It is also true that the expansion of the use of naturalness followed a development of mathematics toward abstraction and specialization. It is our understanding that, as in an attempt to bring to a more concrete and common ground the results of a field, the attribution of naturalness intends to stabilize aspects of mathematical work that fall short of an intuitive treatment. For this reason we believe that, contrary to the reference to nature that this notion explicitly brings with it, the attribution of naturalness manifests a prescriptive component that on the one hand aims at inverting the process of abstraction in favor of a more direct link with mathematical reality, while on the other hand relays on an habit of working with specific mathematical tools that can be acquired and intended only by people working in a specific field. Indeed, a fundamental character of naturalness in mathematics, as argued in San Mauro and Venturi [2015], is its dynamical character, that is, its variance in time: a clear indication of a substantial departure from the static meaning of natural as referent to nature.

In other words, although the reference to nature seems to rely on the acceptance of a realist framework, the use of naturalness judgments does not consist in the recognition of a descriptive character of a piece of mathematics, but instead in the prescriptive attribution of relevance with respect to a given theoretical context and against other pieces of mathematics, that may be thought to be equally relevant.

The role of context then plays a fundamental role in the attribution of naturalness. As a matter of fact calling something natural has the effect of specifying a point of view with respect to a subject matter, whenever the latter has degrees of freedom that allow for different clarifications. The prescriptive character of this act has the effect of making explicit which are the relevant components of a subject matter, and the act is performed precisely when a clear statement in this direction is needed; and this often happens when the abstractness of a field makes difficult the use of intuitive considerations. Moreover, the role of context acts also in determining the scope of naturalness. It is then not surprising that the increase of the reference of natural components in mathematics goes hand in hand with a specialization of the disciplines. Indeed, the smaller and more disconnected the particular scientific communities of working mathematicians are, the more natural a piece of mathematics will seem to a small group of researchers with a common background and working on similar problems.

Without entering here problems of sociology of mathematics, we only want to stress the role that the context plays in the recognition of naturalness. Hence, admitting a general framework in which recognition of natural components is meaningful (i.e. the ideas with respect to which a naturalness judgment may be attributed), the use of the notion of naturalness in mathematics brings with it prescriptive and historical (or contextual) components that, more than describing natural kinds, are meant to specify general ideas with respect to intentions and aims of mathematicians.

Now, coming back to the problem of naturalness of new axioms in set theory, the defects of the intrinsic-extrinsic dichotomy showed that an effective strategy of justification should not presuppose a completely specified concept of set or, respectively, a completely determined reality pointing to the same conceptual counterpart – if we want to save the value of the justification process – and, hence, that the reasons for accepting new axioms cannot be their descriptive character. On the contrary we have seen different pragmatic and historical reasons for justifying new fruitful principles, like $\mathrm{AD}^{L(\mathbb{R})}$, that are meant to specify which are the relevant aspects of set-theoretical reality. The importance of contextual reasons is even more relevant in set theory, where the formal component is accompanied by the presence of an intended interpretation. In other words, in the axiomatization of set theory there are non-mathematical components that play a role in the attribution of meaning to formal sentences.

To summarize our view, we believe that a strategy of justification of new axioms should take into account three different aspects of formalization. At the lowest level we find the formal theory, where purely formal methods can be used to ascertain properties like consistency. At a different level we find the conceptual level, where the concept of set lays (i.e. the semantic counterpart of the formal theory). Contrary to conceptual realism, we believe that this level is not an independent realm where concepts are completely specified kinds that we may describe with axioms. Indeed we maintain a position that accepts the presence, also in mathematics, of open-concepts: entities with degrees of freedom open to further specifications[24]. At a third level we find general ideas that inform mathematical practice. Contrary to the conceptual level, it is here that we find the human component of mathematical work, able to connect syntax and semantics and to specify open concepts with respect to practice. Hence the naturalness of an axiom is to be found when there is accordance between the formal level and the ideal level. The ability of an axiom to capture general ideas that we find in our practice is then able to constitute the meaning of the axiom and thus to influence its semantical counterpart, specifying an open concept[25]. In other words, axioms are called natural when they are able to formalize our scientific practices and, in turn, to modify the basic concept(s) of a theory. However the connections between the conceptual and the ideal level is active also in another direction. Indeed, the realist objectivization of a concept that acts in mathematical practice is able to modify the practice itself. A clear example is given by the effects that the iterative conception had on set-theoretical practice. The habit of thinking about sets as "sets laying in a cumulative hierarchy" has not only become a tacit thesis of set theory, but has suggested new axioms in terms of reflections principle able to describe the indescribability of the class of all sets. This dialectical movement between general ideas coming from mathematical practice, consistent axioms and open concepts lays at the heart of the development of set theory.

What we propose here is, thus, the analysis of the use of naturalness as an indicator of the need for a new justification strategy and the notion of open concepts as constituting a frame to overcome the static character of conceptual realism and the connected intrinsic-extrinsic dichotomy. The notion of naturalness indeed points at the presence of a moving target (as it is the case with the historical concept of set), without proposing stable criteria whose fixity may soon become obsolete, but suggesting a general accordance between theoretical aims and mathematical practice. Moreover, we believe that the notion of open concepts is also able to overcome the limits given by a justification strategy too

[24] This notion has been introduced by Waismann in a series of papers (see [Weismann, 1951a] and [Weismann, 1951b]) in the context of natural language. It has then been argued, in [Shapiro, 2006] that this notion is present and relevant also in a mathematics; the main example being the concept of function.

[25] Although we do not exclude the possibility of a concept revision, the history tells us that this happens only when contradictions have a logical character (i.e., contradiction as in the case of Russell's Paradox) and not mathematical. Consider the example of AC and AD and the fact that their incompatibility did not have the consequence of considering AC or AD false.

often linked to a general concept of sets. Indeed the connection that the naturalness of an axiom manifests between formalization and mathematical practice does not necessarily rest on the recognition of fundamental properties of an alleged general concept of set, but may depends on a particular practice and on knowledge of a local notion of set (i.e. the conceptual counterpart of a collection of sets lying in an initial segment of the cumulative hierarchy).

Now, following our understanding of the use of the notion of naturalness in mathematics, the main question we should ask for the acceptance of set-theoretical axiom is the following: "with respect to which ideas, relevant for the historical development of set theory, for the aim of its formalization and its current practice, we may propose argument in favor of the naturalness of an axiom?"[26].

Calling to the fore history, we need to make clear what we refer to. Since we do not intend here to sketch a historical picture of the development of set theory – and since [Ferreirós, 1999] is a very good reference on this subject – we defer to the following quote the task of giving a rough idea both of the presence of two main periods in the history of axiomatic set theory and of the individuation of the reasons that motivated the development of set theory as a foundational theory.

> Such a depiction [of set theory as a theory with no substantial antecedences] seems suitable for the matatheoretic period that set theory as a field lived from about 1950, [...] but *not* for the more properly theoretical and axiomatic period that antedated 1940. It was in this period, 1904 to 1940, that the core of understanding was gained of set theory, its axiomatic underpinning, the universe V

[26]We can find an antecedent of this perspective in [Hauser, 2004], although in connection with a radically different proposal: the use of a phenomenological standpoint for inquiring the human component of the reasons we adopt for choosing new axioms in set theory. We agree with Hauser that "we must abandon the one-sided view that the objective is something entirely alien to the subjective and that it ought to be studied with complete disregard of the mental life of the mathematician" ([Hauser, 2004], p. 112), but we think that the task of contemporary philosophy of mathematics is not to depict a new context where to argue in favor of the evidence of new axioms, but to make explicit the theoretical intentions of the proponents of new principles and discuss their accordance with the main theoretical ideas that motivate the formalization of set theory. Nonetheless, it is also interesting to note that the phenomenological proposal of [Hauser, 2004] is intended, similarly to ours, to elucidate the natural component of new axioms: "Rather one must examine how the mathematician is intentionally related to those facts because it is on these grounds that he accepts certain abstractions and idealizations as 'natural' or 'reasonable'." ([Hauser, 2004], p. 113). In some sense closely to our historical approach, in [Hauser, 2013] the author attempts a justification of strong axioms of infinity with respect to the metaphysical and theological ideas behind Cantor's theory of the Absolute.

> [...] basically a matter of *understanding and clarifying the concepts of number and function.*[27]

The history of set theory after Cohen's results consists mainly in the development of an independent mathematical field, with its internal motivations. When viewing set theory by this perspective, the extension of ZFC is meant to pursuit the foundation of set theory considered, mostly, as the study of the different possible models of ZFC. Hence independent set-theoretical principles are either studied in order to understand their behavior in different models of ZFC, as happens for example in Hamkins's axiomatic treatment of the multiverse (see [Hamkins, 2012]), or with the aim of selecting the right models among the many possible universes of sets, as Arrigoni-Friedman's Hyperuniverse's Program (see [Arrigoni and Friedman, 2013]) - among others proposals - aims to do.

Nevertheless, when we think of set theory as a foundation of mathematics, we should consider the theory that was studied before 1963, through the lens of the techniques and results developed after 1963. As argued in [Ferreirós, 2011], the role of set theory in clarifying the foundations of mathematics consisted in an attempt to develop a mathematical treatment of the most general notions of number and functions, in terms of the more primitive notion of *arbitrary set*[28]. Arbitrary sets are sets whose existence is independent from our possibility to define them. Therefore, an arbitrary set does not admit, by definition, a precise characterization or an explicit description, but its existence follows from some existential theorems like Cantor's on the uncountability of the real numbers. As a matter of fact, since our language is countable there will always be real numbers whose definition transcends the expressive power of our language. The conception of mathematical objects that is connected with this notion is *quasi combinatorialism*, as described in [Bernays, 1983].

> But analysis is not content with this modest variety of platonism [to take the collection of all numbers as given]; it reflects it to a stronger degree with respect to the following notions: set of numbers, senquece of numbers, and function. It abstracts from the possibility of giving definitions of sets, sequences, and functions. These notion are used in a 'quasi-combinatorial' sense, by which I mean: in the sense of an analogy of the infinite to the finite.

[27] [Ferreirós, 2011], p. 362.

[28] For reasons of space, we decide here not to discuss the difference between the idea of arbitrary set and the iterative conception of set. Although related, we believe the idea of arbitrary set to be wider then the iterative conception and more apt to play a regulative role with respect to the open concept of set. Nonetheless we acknowledge the importance of this theoretical clarification and we plan to elucidate this point in a future work.

> Consider, for example, the different functions which assign to each member of the finite series 1, 2, ..., n a number of the same series. There are n^n functions of this sort, and each of them is obtained by n independent determinations. Passing to the infinite case, we imagine functions engendered by an infinity of independent determinations which assign to each integer an integer, and we reason about the totality of these functions.
>
> In the same way, one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded. We add to this the idea of the totality of these sets. Sequences of real numbers and sets of real numbers are envisaged in an analogous manner. From this point of view, constructive definitions of specific functions, sequences, and sets are only ways to pick out an object which exists independently of, and prior to, the construction.[29]

In [Ferreirós, 2011] there is an attempt to analyze which axioms of ZFC are able to formalize - and to what extent - the notion of arbitrary set, finding that most of them, with the exception of the Axiom of Choice, are very poor in capturing this notion. For the same reason we can easily discard $V = L$ as unnatural, since the restriction given by considering only constructible sets is exactly orthogonal to the notion of arbitrary sets. The fact that in the constructible universe AC holds, nonetheless, is not a hint of its ability to capture this specific notion, but exactly the opposite: the absence of arbitrariness that, as in the finite case, makes choice trivial.

We would like to conclude with a plan for future work. A class of axioms that may be analyzed in this setting is that of Forcing Axioms and, in fact, we plan to inquire their naturalness in a future work. The reason for choosing this type of axioms and not, for example, large cardinal axioms is firstly due to the absence in the literature of a sufficiently philosophical justification of these principles. Moreover, its specific character of local axioms represents a challenge for their justification that would not be perceivable in the case of large cardinals.

Concretely, we propose to reformulate the question about the naturalness of Forcing Axioms in the following way.

1. **Question 1:** Is the notion of arbitrary sets necessary for an intuitive motivation of Forcing Axioms?

[29] [Bernays, 1983], p. 264.

2. **Question 2:** To what extent Forcing Axioms capture and sharpen this idea?

We believe that only giving a positive answers to the above questions we can argue in favor of the naturalness of these axioms. Indeed, following our idea that naturalness judgments hide a prescriptive component, we believe that first we should individuate the relevant aspects of set theory that we intend to formalize, and only subsequently we may argue in favor of the naturalness of an axiom, in terms of its pertinence with the goals of axiomatization – in this particular case the formalization of the notion of arbitrary set.

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